

Introduction to Logic

Propositional Analysis

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Syntax and Semantics

Syntax of Propositional Logic

$\neg p$
 $(p \wedge q)$
 $(p \vee q)$
 $(p \Rightarrow q)$
 $(p \Leftrightarrow q)$

Semantics of Propositional Logic

ϕ	$\neg \phi$	ϕ	ψ	$\phi \wedge \psi$	ϕ	ψ	$\phi \vee \psi$	ϕ	ψ	$\phi \Rightarrow \psi$	ϕ	ψ	$\phi \Leftrightarrow \psi$
T	F	T	T	T	T	T	T	T	T	T	T	T	T
F	T	T	F	F	T	F	T	T	F	F	T	F	F
		F	T	F	F	T	T	F	T	T	F	T	F
		F	F	F	F	F	F	T	F	F	F	F	T

Evaluation versus Satisfaction

Evaluation:

$$\begin{array}{lcl} p^i & = & T \\ q^i & = & F \end{array} \xrightarrow{\text{unique}} \begin{array}{lcl} (p \vee q)^i & = & T \\ (\neg q)^i & = & T \end{array}$$

Satisfaction:

$$\begin{array}{lcl} (p \vee q)^i & = & T \\ (\neg q)^i & = & T \end{array} \xrightarrow[\text{unique}]{\text{not necessarily}} \begin{array}{lcl} p^i & = & T \\ q^i & = & F \end{array}$$

Programme for Today

Properties of Sentences

Validity, Contingency, Unsatisfiability
Satisfiability and Falsifiability

Relationships between Sentences

Equivalence, Entailment, Consistency

Useful "Metatheorems"

Equivalence, Unsatisfiability, Deduction, Consistency
Substitution, Monotonicity, Ramification

Properties of Sentences

Truth Tables

A *truth table* is a table of all possible interpretations for the propositional constants in a language.

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

One column per constant.

One row per interpretation.

For a language with n constants,
there are 2^n interpretations.

Example

Constants			Premises
p	q	r	$p \& q \Rightarrow r$
1	1	1	1
1	1	0	0
1	0	1	1
1	0	0	1
0	1	1	1
0	1	0	1
0	0	1	1
0	0	0	1

Using truth table we can see how a sentence's value as a function of the propositional interpretation.

Oddities

Constants			Premises
p	q	r	$p \& q \Rightarrow r \vee \neg r$
1	1	1	1
1	1	0	1
1	0	1	1
1	0	0	1
0	1	1	1
0	1	0	1
0	0	1	1
0	0	0	1

Constants			Premises
p	q	r	$p \vee (q \vee \neg p) \Rightarrow r \& \neg r$
1	1	1	0
1	1	0	0
1	0	1	0
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0

Occasionally, we find sentences that are odd in that their sentential value does not depend on the values of the propositional constants at all; some sentences are always true and some are always false.

Properties of Sentences

Valid
Contingent
Unsatisfiable

A sentence is *valid* if and only if *every* interpretation satisfies it.

A sentence is *contingent* if and only if *some* interpretation satisfies it and *some* interpretation falsifies it.

A sentence is *unsatisfiable* if and only if *no* interpretation satisfies it.

The upshot is that we can classify sentences based on this property.

Properties of Sentences

Valid	}	A sentence is <i>satisfiable</i> if and only if it is either valid or contingent.
Contingent		
Unsatisfiable	}	A sentence is <i>falsifiable</i> if and only if it is contingent or unsatisfiable.

Possible Worlds

Constants			Premises		
p	q	r	$p \Rightarrow q \ \& \ r \Rightarrow r$	$\neg r$	
1	1	1	1	0	
1	1	0	0	1	
1	0	1	0	0	
1	0	0	0	1	
0	1	1	1	0	
0	1	0	1	0	
0	0	1	1	0	
0	0	0	1	1	



Contingent sentences decrease the number of possible worlds.

Possible Worlds

Constants			Premises		
p	q	r	$p \Rightarrow q \ \& \ r \Rightarrow r$	$\neg r$	
1	1	1	1	0	
1	1	0	0	1	
1	0	1	0	0	
1	0	0	0	1	
0	1	1	1	0	
0	1	0	1	0	
0	0	1	1	0	
0	0	0	1	1	



Contingent sentences decrease the number of possible worlds.

Possible Worlds

Constants			Premises		
p	q	r	$p \Rightarrow q \ \& \ r \Rightarrow q \Rightarrow r$	$\neg r$	
1	1	1	1	1	0
1	1	0	0	0	1
1	0	1	0	1	0
1	0	0	0	1	1
0	1	1	1	1	0
0	1	0	1	0	1
0	0	1	1	1	0
0	0	0	1	1	1



Contingent sentences decrease the number of possible worlds.

Possible Worlds

Constants			Premises		
p	q	r	$p \Rightarrow q \ \& \ r \Rightarrow q \Rightarrow r$	$\neg r$	
1	1	1	1	1	0
1	1	0	0	0	1
1	0	1	0	1	0
1	0	0	0	1	1
0	1	1	1	1	0
0	1	0	1	0	1
0	0	1	1	1	0
0	0	0	1	1	1



Contingent sentences decrease the number of possible worlds. Ideally, when we have enough contingent sentences, we know everything about the world.

Possible Worlds

Constants			Premises
p	q	r	$(p \Rightarrow q \ \& \ r) \vee \neg(p \Rightarrow q \ \& \ r)$
1	1	1	1
1	1	0	1
1	0	1	1
1	0	0	1
0	1	1	1
0	1	0	1
0	0	1	1
0	0	0	1

Note that, by contrast, valid sentences tell us nothing; they do not eliminate any worlds.

Possible Worlds

Constants			Premises
p	q	r	$(p \supset q \ \& \ r) \ \& \ \neg(p \supset q \ \& \ r)$
1	1	1	0
1	1	0	0
1	0	1	0
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0

And unsatisfiable sentences ruin everything.

Valid Equivalences

Double Negation:

$$p \Leftrightarrow \neg \neg p$$

deMorgan's Laws:

$$\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$$

$$\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$$

Implications:

$$(p \Rightarrow q) \Leftrightarrow (\neg p \vee q)$$

Biconditionals:

$$(p \Leftrightarrow q) \Leftrightarrow ((p \Rightarrow q) \wedge (q \Rightarrow p))$$

Be that as it may, it turns out that, from the point of view of reasoning, non-contingent sentences are interesting. As we shall see, we can exploit these properties in logical reasoning. For example, valid equivalences allow us to rewrite sentences by substituting equivalent subexpressions.

Valid Implications

Implication Introduction:

$$p \Rightarrow (q \Rightarrow p)$$

Implication Distribution

$$(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$$

Implication Reversal

$$(\neg q \Rightarrow \neg p) \Rightarrow (p \Rightarrow q)$$

And, as we shall see, valid implications allow us to derive conclusions from premises.

Relationships Between Sentences

Comparison of Sentences

Constants		Premises	
p	q	$p \Rightarrow q$	$\neg p \vee q$
1	1	1	1
1	0	0	0
0	1	1	1
0	0	1	1

Logical Equivalence

A sentence ϕ is *logically equivalent* to a sentence ψ if and only they have the same value for every propositional interpretation.

$(p \Rightarrow q)$ is logically equivalent to $(\neg p \vee q)$
 p is logically equivalent to $\neg \neg p$

$(p \wedge q)$ is *not* logically equivalent to $(p \vee q)$

Another Comparison of Sentences

Constants		Premises	
p	q	p & q	p ! q
1	1	1	1
1	0	0	1
0	1	0	1
0	0	0	0

Logical Entailment

A premise ϕ *logically entails* a conclusion ψ (written as $\phi \models \psi$) if and only if every interpretation that satisfies ϕ also satisfies ψ .

$$(p \wedge q) \models (p \vee q)$$

$$p \models (p \vee q)$$

$$(p \wedge q) \models p$$

$$p \not\models (p \wedge q)$$

Most important concept today.
Go back to table for examples.

Logical Entailment $\neq$ Logical Equivalence

$$p \models (p \vee q)$$

$$(p \vee q) \not\models p$$

Analogy in arithmetic: inequalities rather than equations

Sets of Premises

A set of premises Δ *logically entails* a conclusion φ (written as $\Delta \models \varphi$) if and only if every interpretation that satisfies *all* of the premises also satisfies the conclusion.

$$\{p, q\} \models (p \wedge q)$$

Sets of Conclusions

A premise φ *logically entails* a set of conclusions if and only if every interpretation that satisfies the premise satisfies *all* of the conclusions.

$$(p \wedge q) \models \{p, q\}$$

Validities

If $\{\} \models \varphi$, then φ is valid.

Examples:

$$\{\} \models p \vee \neg p$$

$$\{\} \not\models p$$

$$\{\} \not\models p \wedge \neg p$$

The empty set of premises is satisfied by every interpretation. Consequently, if it entails a sentence, that sentence must be true in every interpretation, i.e. it is valid.

Vacuity

If Δ is unsatisfiable, then $\Delta \models \varphi$ for *all* φ !

Examples:

$$\{p, \neg p\} \models p$$

$$\{p, \neg p\} \models \neg p$$

$$\{p, \neg p\} \models q$$

By definition, an unsatisfiable set of sentences is not satisfied by *any* interpretation. Consequently, it is trivially true that every interpretation that satisfies that set satisfies every sentence.

Unsatisfiable assumptions entail everything!!!

Third Comparison of Sentences

Constants		Premises	
p	q	$p \wedge q$	$\neg p \wedge \neg q$
1	1	1	0
1	0	1	1
0	1	1	1
0	0	0	1

→
→

Logical Consistency

A sentence ϕ is *consistent with* a sentence ψ if and only if there is a truth assignment that satisfies both ϕ and ψ .

p is logically consistent with q

$(p \vee q)$ is logically consistent with $(\neg p \vee \neg q)$

$(p \Rightarrow q)$ is logically consistent with $(\neg p \vee q)$

p is *not* consistent with $\neg p$

Is $(p \wedge \neg p)$ logically consistent with $(p \wedge \neg p)$?

Is $(p \wedge \neg p)$ logically equivalent to $(p \wedge \neg p)$?

Courtroom analogy.

Unsatisfiable sentences are not consistent with anything.

Metatheorems

Propositional Metatheorems

*A metatheorem is a theorem **about** logic.*

Monotonicity Theorem
Ramification Theorem

Equivalence Theorem
Substitution Theorem
Deduction Theorem

Unsatisfiability Theorem
Consistency Theorem

A propositional metatheorem is a theorem *about* propositional logic (as opposed to theorems about circuits or whodunnits).

Monotonicity Theorem

If $\Gamma \models \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \models \varphi$.

Example: $\{p, q\} \models p \wedge q$
Therefore $\{p, q, r\} \models p \wedge q$

The more you know, the more is entailed.

Ramification Theorem

If $\Omega \models \Delta$ and $\Gamma \subseteq \Delta$, then $\Omega \models \Gamma$.

Example: $\{p \wedge q\} \models \{p, q\}$

Therefore $\{p \wedge q\} \models \{p\}$

If you can conclude more, you can conclude less.

Equivalence Theorem

Theorem: A sentence ϕ and a sentence ψ are *logically equivalent* if and only if the sentence $(\phi \leftrightarrow \psi)$ is *valid*.

$\neg(p \wedge q)$ is logically equivalent to $(\neg p \vee \neg q)$

if and only if

$(\neg(p \wedge q) \leftrightarrow (\neg p \vee \neg q))$ is valid

Upshot: We can determine equivalence of sentences by checking validity of a single sentence.

Upshot: We can demonstrate validity of a biconditional by checking equivalence of the constituents.

Equivalence Theorem

Constants		Premises		Conclusions
p	q	$\neg(p \& q)$	$\neg p \vee \neg q$	$\neg(p \& q) \leftrightarrow \neg p \vee \neg q$
1	1	0	0	1
1	0	1	1	1
0	1	1	1	1
0	0	1	1	1

Substitution Theorem

Let $\chi_{\varphi \leftarrow \psi}$ stand for a copy of χ where zero or more occurrences of φ have been replaced by ψ .

Example: Let $\chi = (\neg \neg p \vee q)$, then $\chi_{\neg \neg p \leftarrow p} = (p \vee q)$.

Substitution Theorem: If $(\varphi \Leftrightarrow \psi)$ is valid, then the sentence $\chi_{\varphi \leftarrow \psi}$ is logically equivalent to χ .

Example: Since $(p \Leftrightarrow \neg \neg p)$ is valid, we know that the sentence $(\neg \neg p \vee q)$ is logically equivalent to $(p \vee q)$.

Substitution Example

Babbage Show Instructions

Premises:

$\neg p \mid q$
 $p \mid q$

Truth Table

Constants		Premises	
p	q	$\neg p \mid q$	$p \mid q$
1	1	1	1
1	0	1	1
0	1	1	1
0	0	0	0

Deduction Theorem

Theorem: A sentence ϕ *logically entails* a sentence ψ if and only if $(\phi \Rightarrow \psi)$ is *valid*.

More generally, a finite set of sentences $\{\phi_1, \dots, \phi_n\}$ logically entails ϕ if and only if the compound sentence $(\phi_1 \wedge \dots \wedge \phi_n \Rightarrow \phi)$ is valid.

Is $((p \Rightarrow q) \wedge (m \Rightarrow p \vee q) \Rightarrow (m \Rightarrow q))$ valid?
 $\{(p \Rightarrow q), (m \Rightarrow p \vee q)\} \models (m \Rightarrow q)$?

Upshot: We can determine logical entailment between sentences by checking validity of a single sentence. *And vice versa*.

Deduction Theorem

$\{(m \Rightarrow p \vee q), (p \Rightarrow q)\} \models (m \Rightarrow q)$?

Is $((m \Rightarrow p \vee q) \wedge (p \Rightarrow q) \Rightarrow (m \Rightarrow q))$ valid?

Constants			Premises			Conclusions
m	p	q	$m \Rightarrow p \vee q$	$p \Rightarrow q$	$m \Rightarrow q$	$(m \Rightarrow p \vee q) \wedge (p \Rightarrow q) \Rightarrow (m \Rightarrow q)$
1	1	1	1	1	1	1
1	1	0	1	0	0	1
1	0	1	1	1	1	1
1	0	0	0	1	0	1
0	1	1	1	1	1	1
0	1	0	1	0	1	1
0	0	1	1	1	1	1
0	0	0	1	1	1	1

Unsatisfiability Theorem

Theorem: $\Delta \models \varphi$ if and only if $\Delta \cup \{\neg \varphi\}$ is unsatisfiable.

Proof: Suppose that $\Delta \models \varphi$. If an interpretation satisfies Δ , then it must also satisfy φ . But then it cannot satisfy $\neg \varphi$. Therefore, $\Delta \cup \{\neg \varphi\}$ is unsatisfiable.

Suppose that $\Delta \cup \{\neg \varphi\}$ is unsatisfiable. Then every interpretation that satisfies Δ must *fail* to satisfy $\neg \varphi$, i.e. it must satisfy φ . Therefore, $\Delta \models \varphi$.

Upshot: We can determine logical entailment between sentences by checking unsatisfiability of a set of sentences.

Translation: Assume false and show contradiction.

Consistency Theorem

Theorem: A sentence ϕ is logically *consistent* with a sentence ψ if and only if the sentence $(\phi \wedge \psi)$ is *satisfiable*. More generally, a sentence ϕ is logically consistent with a finite set of sentences $\{\phi_1, \dots, \phi_n\}$ if and only if the compound sentence $(\phi_1 \wedge \dots \wedge \phi_n \wedge \phi)$ is satisfiable.

Is $(p \vee q)$ consistent with $(\neg p \vee \neg q)$?

Is $((p \vee q) \wedge (\neg p \vee \neg q))$ satisfiable?

Upshot: We can determine consistency of sentences by checking satisfiability of a single sentence.

Metareasoning

CS 157 Quiz Question #1

Is the sentence $(p \Rightarrow q) \Leftrightarrow (p \Rightarrow (q \Rightarrow p))$ valid, contingent, or unsatisfiable?

CS 157 Quiz Question #1

Is the sentence $(p \Rightarrow q) \Leftrightarrow (p \Rightarrow (q \Rightarrow p))$ valid, contingent, or unsatisfiable?

p	q	$(p \Rightarrow q) \Leftrightarrow (p \Rightarrow (q \Rightarrow p))$
1	1	1
1	0	0
0	1	1
0	0	1

CS 157 Quiz Question #1

Is the sentence $(p \Rightarrow q) \Leftrightarrow (p \Rightarrow (q \Rightarrow p))$ valid, contingent, or unsatisfiable?

$(p \Rightarrow q)$ is sometimes true and sometimes false.

$(p \Rightarrow (q \Rightarrow p))$ is always true, i.e. it is a valid sentence.

$(p \Rightarrow q)^i = (p \Rightarrow (q \Rightarrow p))^i$ for some i .

$(p \Rightarrow q)^i \neq (p \Rightarrow (q \Rightarrow p))^i$ for some i .

$(p \Rightarrow q) \Leftrightarrow (p \Rightarrow (q \Rightarrow p))$ is contingent.

CS 157 Quiz Question #2

Is the sentence $((p \Leftrightarrow \neg q) \Rightarrow (\neg p \vee \neg q))$ valid, contingent, or unsatisfiable?

CS 157 Quiz Question #2

Is the sentence $((p \Leftrightarrow \neg q) \Rightarrow (\neg p \vee \neg q))$ valid, contingent, or unsatisfiable?

p	q	$((p \Leftrightarrow \neg q) \Rightarrow (\neg p \vee \neg q))$
1	1	1
1	0	1
0	1	1
0	0	1

CS 157 Quiz Question #2

Is the sentence $((p \leftrightarrow \neg q) \Rightarrow (\neg p \vee \neg q))$ valid, contingent, or unsatisfiable?

$$((p \leftrightarrow \neg q) \Rightarrow (\neg p \vee \neg q))$$

CS 157 Quiz Question #2

Is the sentence $((p \leftrightarrow \neg q) \Rightarrow (\neg p \vee \neg q))$ valid, contingent, or unsatisfiable?

$$(p \leftrightarrow \neg q) \Rightarrow (\neg p \vee \neg q)$$

$$(p \leftrightarrow \neg q) \Rightarrow (p \Rightarrow \neg q)$$

Substitution Theorem
and
 $(p \Rightarrow \neg q) \Leftrightarrow (\neg p \vee \neg q)$

CS 157 Quiz Question #2

Is the sentence $((p \leftrightarrow \neg q) \Rightarrow (\neg p \vee \neg q))$ valid, contingent, or unsatisfiable?

$$((p \leftrightarrow \neg q) \Rightarrow (\neg p \vee \neg q))$$

$$(p \leftrightarrow \neg q) \Rightarrow (p \Rightarrow \neg q)$$

$$(p \Rightarrow \neg q) \wedge (\neg q \Rightarrow p) \Rightarrow (p \Rightarrow \neg q)$$

Substitution Theorem
and
 $(p \leftrightarrow \neg q) \Leftrightarrow (p \Rightarrow \neg q) \wedge (\neg q \Rightarrow p)$

CS 157 Quiz Question #2

Is the sentence $((p \leftrightarrow \neg q) \Rightarrow (\neg p \vee \neg q))$ valid, contingent, or unsatisfiable?

$$((p \leftrightarrow \neg q) \Rightarrow (\neg p \vee \neg q))$$

$$(p \leftrightarrow \neg q) \Rightarrow (p \Rightarrow \neg q)$$

$$(p \Rightarrow \neg q) \wedge (\neg q \Rightarrow p) \Rightarrow (p \Rightarrow \neg q)$$

$$(p \Rightarrow \neg q) \wedge (\neg q \Rightarrow p) \models (p \Rightarrow \neg q)$$

Deduction Theorem

CS 157 Quiz Question #2

Is the sentence $((p \leftrightarrow \neg q) \Rightarrow (\neg p \vee \neg q))$ valid, contingent, or unsatisfiable?

$$((p \leftrightarrow \neg q) \Rightarrow (\neg p \vee \neg q))$$

$$(p \leftrightarrow \neg q) \Rightarrow (p \Rightarrow \neg q)$$

$$(p \Rightarrow \neg q) \wedge (\neg q \Rightarrow p) \Rightarrow (p \Rightarrow \neg q)$$

$$(p \Rightarrow \neg q) \wedge (\neg q \Rightarrow p) \models (p \Rightarrow \neg q)$$

$$\{(p \Rightarrow \neg q), (\neg q \Rightarrow p)\} \models (p \Rightarrow \neg q)$$

Definition of Entailment
Definition of Conjunction

CS 157 Quiz Question #2

Is the sentence $((p \leftrightarrow \neg q) \Rightarrow (\neg p \vee \neg q))$ valid, contingent, or unsatisfiable?

$$((p \leftrightarrow \neg q) \Rightarrow (\neg p \vee \neg q))$$

$$(p \leftrightarrow \neg q) \Rightarrow (p \Rightarrow \neg q)$$

$$(p \Rightarrow \neg q) \wedge (\neg q \Rightarrow p) \Rightarrow (p \Rightarrow \neg q)$$

$$(p \Rightarrow \neg q) \wedge (\neg q \Rightarrow p) \models (p \Rightarrow \neg q)$$

$$\{(p \Rightarrow \neg q), (\neg q \Rightarrow p)\} \models (p \Rightarrow \neg q)$$

$$\{(p \Rightarrow \neg q)\} \models (p \Rightarrow \neg q)$$

Monotonicity Theorem

CS 157 Quiz Question #3a

Let Γ and Δ be arbitrary sets of sentences.
Let φ be an arbitrary sentence.
If $\Gamma \models \varphi$ and $\Delta \models \varphi$, does $\Gamma \cup \Delta \models \varphi$?

CS 157 Quiz Question #3a

Let Γ and Δ be arbitrary sets of sentences.
Let φ be an arbitrary sentence.
Is $\Gamma \models \varphi$ and $\Delta \models \varphi$, does $\Gamma \cup \Delta \models \varphi$?

Let Γ be $\{p\}$ and Δ be $\{q\}$ and φ be $(p \vee q)$.

Obviously $\{p\} \models p \vee q$

Obviously $\{q\} \models p \vee q$

But $\{p\} \cup \{q\} = \{p, q\}$ and $\{p, q\} \models p \vee q$

Does this work for all Γ and Δ and φ ?
Yes, by Monotonicity Theorem.

CS 157 Quiz Question #3b

Let Γ and Δ be arbitrary sets of sentences.
Let φ be an arbitrary sentence.
Is $\Gamma \models \varphi$ and $\Delta \models \varphi$, does $\Gamma \cap \Delta \models \varphi$?

CS 157 Quiz Question #3b

Let Γ and Δ be arbitrary sets of sentences.

Let φ be an arbitrary sentence.

Is $\Gamma \models \varphi$ and $\Delta \models \varphi$, does $\Gamma \cap \Delta \models \varphi$?

Let Γ be $\{p\}$ and Δ be $\{q\}$ and φ be $(p \vee q)$.

Obviously $\{p\} \models p \vee q$

Obviously $\{q\} \models p \vee q$

But $\{p\} \cap \{q\} = \{\}$ and $\{\} \not\models p \vee q$.

Answer to our question: No.

Tools

Course Website

<http://logica.stanford.edu>

Babbage
Show Instructions

Premises:

$((p \leftrightarrow \neg q) \Rightarrow (\neg p \mid \neg q))$

Truth Table

Constants		Premises
p	q	$(p \leftrightarrow \neg q) \Rightarrow \neg p \mid \neg q$
1	1	1
1	0	1
0	1	1
0	0	1

Enter our three sentences - $((p \leftrightarrow \neg q) \Rightarrow (\neg p \mid \neg q))$

Quine

Undo Redo Help

$(p \leftrightarrow \neg q) \Rightarrow (\neg p \mid \neg q)$

Enter a replacement:

$p \Rightarrow \neg q$

Add
Cancel

Press the escape key to enter edit mode. Use the arrow keys to navigate. Press escape key again to exit.

Replace	Negation Introduction	Implication Introduction	Universal Introduction
	Negation Elimination	Implication Elimination	Universal Elimination
	Negation Distribution	Contrapositive	Quantifier Reversal
	Negation Extraction	Biconditional Introduction	Quantifier Distribution
	Reorder	Biconditional Elimination	Variable Renaming
Distribute			

Enter $p \& q \Rightarrow r$
 IE $\sim(p \& q) \mid r$
 Focus on $\sim(p \& q)$
 Negation Distribution $(\sim p \mid \sim q) \mid r$
 Defocus
 Distribute $\sim p \mid (\sim q \mid r)$
 Focus $(\sim q \mid r)$
 Implication Introduction
 Defocus
 Implication Introduction



Rotors