

Introduction to Logic

Direct Proofs

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Logical Entailment

A set of premises Δ logically entails a conclusion φ (written $\Delta \models \varphi$) if and only if every interpretation that satisfies Δ also satisfies φ .

Semantic Reasoning (Truth Tables)

Premises			Conclusion		
m	p	q	m	p	q
T	T	T	T	T	T
T	T	F	T	T	F
T	F	T	T	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	T	F	F	T	F
F	F	T	F	F	T
F	F	F	F	F	F

With n constants, there are 2^n truth assignments.

In the case of Propositional Logic, we can check logical entailment by examining a truth table for the language. Checking logical entailment with truth tables has the merit of being conceptually simple. However, it is not always the most practical method. The number of truth assignments of a language grows exponentially with the number of logical constants. When the number of logical constants in a propositional language is large, it may be impossible to process its truth table.

Symbolic Manipulation (Metatheorems)

Is the sentence $((p \leftrightarrow \neg q) \Rightarrow (\neg p \vee \neg q))$ valid, contingent, or unsatisfiable?

$$(p \leftrightarrow \neg q) \Rightarrow (\neg p \vee \neg q)$$

$$(p \leftrightarrow \neg q) \Rightarrow (p \Rightarrow \neg q)$$

$$(p \Rightarrow \neg q) \wedge (\neg q \Rightarrow p) \Rightarrow (p \Rightarrow \neg q)$$

$$(p \Rightarrow \neg q) \wedge (\neg q \Rightarrow p) \models (p \Rightarrow \neg q)$$

$$\{(p \Rightarrow \neg q), (\neg q \Rightarrow p)\} \models (p \Rightarrow \neg q)$$

$$\{(p \Rightarrow \neg q)\} \models (p \Rightarrow \neg q)$$

Monotonicity Theorem

Smattering of metatheorems.

Symbolic Manipulation (Proofs)

Truth Tables:

Guaranteed to succeed

Often quite large

Problem Rewriting using metatheorems:

Frequently less work than truth tables

More intuitive than truth tables

Sometimes works, sometimes don't

Theorem Proving:

Frequently less work than truth tables

More intuitive than truth tables

Always works. Systematic.

Proof methods provide an alternative way of checking logical entailment that addresses this problem. In many cases, it is possible to create a proof of a conclusion from a set of premises that is much smaller than the truth table for the language; moreover, it is often possible to find such proofs with less work than is necessary to check the entire truth table.

Proof Systems

Popular Types of Proof Systems:

- Direct Proofs (Hilbert)
- Natural Deduction (Fitch)
- Refutation proofs (Resolution)

Others:

Gentzen Systems
Sequent Calculi
and so forth

Over the years, logicians have proposed a variety of different proof "systems". In direct proof systems (which are sometimes called Hilbert systems), proofs take the form of sequences of sentences leading from premises to conclusions. In hypothetical reasoning systems, proofs have a tree-like structure - with subproofs nested within superproofs. Both linear proofs systems and hypothetical proof systems are "direct" or "generative" in that they start with premises and end with conclusions. In refutation systems, we start with premises and the negations of our conclusions and derive contradictions. Resolution theorem proving is the most common system of this sort.

Note that these are not the only proof systems. There are also Gentzen systems and sequent calculi, and so forth. These other types of proof systems are theoretically interesting. However, they are less useful in producing proofs in practical situations, and so we will not talk about them here.

On the other hand, we will go into detail on the first three types of proof systems. Today, we will talk about linear proofs. In our next session, we will look at Hypothetical proofs and the Fitch proof system. In the session after that, we will look at propositional resolution. As we will see, each of these systems is sound and complete for propositional logic, i.e. it produces only correct answers and each by itself is sufficient to demonstrate logical entailment. None is best in all cases, and it is common in practice to combine these forms of reasoning. However, for pedagogical purposes, we will treat them independently in this course.

Rules of Inference

Direct proofs may be familiar from geometry. SSS SAS and so forth.

Schemas / Schemata

A *pattern / schema* is an expression satisfying the grammatical rules of our language except for the occurrence of *metavariables* (written here as Greek letters) in place of various subparts of the expression.

For example, the following expression is a schema with metavariables ϕ and ψ .

$$\phi \Rightarrow (\psi \Rightarrow \phi)$$

*NB: Metavariables are not symbols in our language; they are symbols we use in talking **about** our language.*

A schema is an expression satisfying the grammatical rules of our language except for the occurrence of metavariables (written here as Greek letters) in place of various subparts of the expression. For example, the following expression is a schema with metavariables ϕ and ψ .

Rules of Inference

A *rule of inference* is a pattern of reasoning consisting of zero or more schemas, called *premises*, and one or more additional schemas, called *conclusions*.

$$\frac{\phi \Rightarrow \psi \quad \phi}{\psi}$$

This rule is *Implication Elimination* or *Modus Ponens*.

A rule of inference is a pattern of reasoning consisting of some schemas, called premises, and one or more additional schemas, called conclusions. Rules of inference are often written as shown below. The schemas above the line are the premises, and the schemas below the line are the conclusions. The rule in this case is called Implication Elimination (or IE), because it eliminates the implication from the first premise.

Rules of Inference for $\neg$ and $\Rightarrow$

Implication Elimination

$$\frac{\phi \Rightarrow \psi \quad \phi}{\psi}$$

Implication Creation

$$\frac{\psi}{\phi \Rightarrow \psi}$$

Implication Distribution

$$\frac{(\phi \Rightarrow (\psi \Rightarrow \chi))}{(\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi)}$$

Implication Reversal

$$\frac{\neg \psi \Rightarrow \neg \phi}{\phi \Rightarrow \psi}$$

Implication Creation (IC), shown here, is another rule of inference. This rule tells us that, if a sentence ψ is true, we can infer $(\phi \Rightarrow \psi)$ for any ϕ whatsoever. Implication Distribution (ID) tells us that implication can be distributed over other implications. If $(\phi \Rightarrow (\psi \Rightarrow \chi))$ is true, then we can infer $((\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi))$.

Rule Instances

An *instance* of a rule of inference is the rule obtained by consistently substituting sentences for the metavariables in the rule.

Rule of Inference:

$$\frac{\phi \Rightarrow \psi \quad \phi}{\psi}$$

Instances:

$$\frac{p \Rightarrow q \quad p}{q}$$

$$\frac{\neg p \Rightarrow q \quad \neg p}{q}$$

$$\frac{(p \Rightarrow q) \Rightarrow (q \Rightarrow r) \quad p \Rightarrow q}{q \Rightarrow r}$$

An instance of a rule of inference is the rule obtained by consistently substituting sentences for the metavariables in the rule. For example, the following is an instance of Implication Elimination.

If a metavariable occurs more than once, the same expression must be used for every occurrence. For example, in the case of Implication Elimination, it would not be acceptable to replace one occurrence of ϕ with one expression and the other occurrence of ϕ with a different expression.

Note that the replacement can be an arbitrary expression so long as the result is a legal expression. For example, in the following instance of Implication Elimination, we have replaced the variables by compound sentences.

Note also that there are infinitely many sentences in our language. Even though we start with finitely many propositional constants (in a propositional vocabulary) and finitely many operators, we can combine them in arbitrarily many ways. The upshot is that there are infinitely many instances of any rule of inference involving metavariables.

Rule Application

Premises:

p
 $(p \Rightarrow q)$
 $(p \Rightarrow q) \Rightarrow (q \Rightarrow r)$

Rule of Inference:

$\phi \Rightarrow \psi$
 ϕ
 ψ

Rule Instance 1:

$p \Rightarrow q$
 p
 q

Rule Instance 2:

$(p \Rightarrow q) \Rightarrow (q \Rightarrow r)$
 $(p \Rightarrow q)$
 $(q \Rightarrow r)$

Conclusion 1:

q

Conclusion 2:

$(q \Rightarrow r)$

A rule applies to a set of sentences if and only if there is an instance of the rule in which all of the premises are in the set. In this case, the conclusions of the instance are the results of the rule application.

For example, if we had a set of sentences containing the sentence p and the sentence $(p \Rightarrow q)$, then we could apply Implication Elimination to derive q as a result. If we had a set of sentences containing the sentence $(p \Rightarrow q)$ and the sentence $(p \Rightarrow q) \Rightarrow (q \Rightarrow r)$, then we could apply Implication Elimination to derive $(q \Rightarrow r)$ as a result.

Top Level Sentences Only!!!

Premises:

$(p \Rightarrow q)$
 $(p \Rightarrow r)$

Rule of Inference:

$\phi \Rightarrow \psi$
 ϕ
 ψ

Rule Instance:

$(p \Rightarrow q)$
 p
 q

Conclusion:

X $(q \Rightarrow r)$ X No, no, no!

Rules cannot be applied to parts of sentences!!!

In using rules of inference, it is important to remember that they apply only to top-level sentences, not to components of sentences. While applying to components sometimes works, it can also lead to incorrect results.

As an example of such a problem, consider the incorrect application of Implication Elimination shown below. Suppose we believe $(p \Rightarrow q)$ and $(p \Rightarrow r)$. We might try to apply Implication Elimination here, taking the first premise as the implication and taking the occurrence of p in the second premise as the matching condition, leading us to conclude $(q \Rightarrow r)$.

Unfortunately, this is not a proper logical conclusion from the premises, as we all know from experience and as we can quickly determine by looking at the associated truth table. It is important to remember that rules of inference apply only to top-level sentences.

Direct Proof

A *direct proof* of a conclusion from a set of premises is a sequence of sentences terminating in the conclusion in which each item is either

- (1) a premise,
- (2) an instance of a valid schema, or
- (3) the result of applying a rule of inference to earlier items in sequence.

A schema is an expression satisfying the grammatical rules of our language except for the occurrence of metavariables (written here as Greek letters) in place of various subparts of the expression. For example, the following expression is a schema with metavariables ϕ and ψ .

Example 1

Given p and $(p \Rightarrow q)$ and $((p \Rightarrow q) \Rightarrow (q \Rightarrow r))$, prove r .

Here is an example. Suppose we have the set of sentences we saw earlier. We start our proof by writing out our premises. We believe p ; we believe $(p \Rightarrow q)$; and we believe that $(p \Rightarrow q) \Rightarrow (q \Rightarrow r)$. Using Implication Elimination on the first premise and the second premise, we derive q . Applying Implication Elimination to the second premise and the third premise, we derive $(q \Rightarrow r)$. Finally, we use the derived premises on lines 4 and 5 to arrive at our desired conclusion.

Example 1

Given p and $(p \Rightarrow q)$ and $((p \Rightarrow q) \Rightarrow (q \Rightarrow r))$, prove r .

- | | |
|--|----------|
| 1. p | Premise |
| 2. $p \Rightarrow q$ | Premise |
| 3. $(p \Rightarrow q) \Rightarrow (q \Rightarrow r)$ | Premise |
| 4. q | IE: 2, 1 |

$\frac{\phi \Rightarrow \psi}{\psi}$	$\frac{\psi}{\phi \Rightarrow \psi}$	$\frac{(\phi \Rightarrow (\psi \Rightarrow \chi))}{(\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi)}$	$\frac{\neg \psi \Rightarrow \neg \phi}{\phi \Rightarrow \psi}$
IE	IC	ID	IR

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| 1. p | Premise |
| 2. $p \Rightarrow q$ | Premise |
| 3. $(p \Rightarrow q) \Rightarrow (q \Rightarrow r)$ | Premise |
| 4. q | IE: 2, 1 |
| 5. $q \Rightarrow r$ | IE: 3, 2 |

$\frac{\phi \Rightarrow \psi}{\psi}$	$\frac{\psi}{\phi \Rightarrow \psi}$	$\frac{(\phi \Rightarrow (\psi \Rightarrow \chi))}{(\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi)}$	$\frac{\neg \psi \Rightarrow \neg \phi}{\phi \Rightarrow \psi}$
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| 2. $p \Rightarrow q$ | Premise |
| 3. $(p \Rightarrow q) \Rightarrow (q \Rightarrow r)$ | Premise |
| 4. q | IE: 2, 1 |
| 5. $q \Rightarrow r$ | IE: 3, 2 |
| 6. r | IE: 5, 4 |

$$\begin{array}{c}
 \frac{\phi \Rightarrow \psi}{\frac{\phi}{\psi}} \quad \frac{\psi}{\phi \Rightarrow \psi} \quad \frac{(\phi \Rightarrow (\psi \Rightarrow \chi))}{(\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi)} \quad \frac{\neg \psi \Rightarrow \neg \phi}{\phi \Rightarrow \psi} \\
 \text{IE} \quad \text{IC} \quad \text{ID} \quad \text{IR}
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Example 2

Given $(p \Rightarrow q)$ and $(q \Rightarrow r)$, prove $(p \Rightarrow r)$.

- | | |
|----------------------|---------|
| 1. $p \Rightarrow q$ | Premise |
| 2. $q \Rightarrow r$ | Premise |

$$\begin{array}{c}
 \frac{\phi \Rightarrow \psi}{\frac{\phi}{\psi}} \quad \frac{\psi}{\phi \Rightarrow \psi} \quad \frac{(\phi \Rightarrow (\psi \Rightarrow \chi))}{(\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi)} \quad \frac{\neg \psi \Rightarrow \neg \phi}{\phi \Rightarrow \psi} \\
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Here is another example. Whenever p is true, q is true. Whenever q is true, r is true. With these as premises, we can prove that, whenever p is true, r is true. On line 3, we use Implication Creation to derive $(p \Rightarrow (q \Rightarrow r))$. On line 4, we use Implication Distribution to distribute the implication in line 3. Finally, on line 5, we use Implication Elimination to produce the desired result.

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- | | |
|--------------------------------------|---------|
| 1. $p \Rightarrow q$ | Premise |
| 2. $q \Rightarrow r$ | Premise |
| 3. $p \Rightarrow (q \Rightarrow r)$ | IC: 2 |

$$\begin{array}{c}
 \frac{\phi \Rightarrow \psi}{\frac{\phi}{\psi}} \quad \frac{\psi}{\phi \Rightarrow \psi} \quad \frac{(\phi \Rightarrow (\psi \Rightarrow \chi))}{(\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi)} \quad \frac{\neg \psi \Rightarrow \neg \phi}{\phi \Rightarrow \psi} \\
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1. $p \Rightarrow q$ Premise
2. $q \Rightarrow r$ Premise
3. $p \Rightarrow (q \Rightarrow r)$ IC: 2
4. $(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$ ID: 3

$$\begin{array}{c}
 \frac{\phi \Rightarrow \psi}{\frac{\phi}{\psi}} \text{ IE} \quad \frac{\psi}{\frac{\phi \Rightarrow \psi}{\phi \Rightarrow \psi}} \text{ IC} \quad \frac{(\phi \Rightarrow (\psi \Rightarrow \chi))}{(\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi)} \text{ ID} \quad \frac{\neg \psi \Rightarrow \neg \phi}{\phi \Rightarrow \psi} \text{ IR}
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2. $q \Rightarrow r$ Premise
3. $p \Rightarrow (q \Rightarrow r)$ IC: 2
4. $(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$ ID: 3
5. $(p \Rightarrow r)$ IE: 4, 1

$$\begin{array}{c}
 \frac{\phi \Rightarrow \psi}{\frac{\phi}{\psi}} \text{ IE} \quad \frac{\psi}{\frac{\phi \Rightarrow \psi}{\phi \Rightarrow \psi}} \text{ IC} \quad \frac{(\phi \Rightarrow (\psi \Rightarrow \chi))}{(\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi)} \text{ ID} \quad \frac{\neg \psi \Rightarrow \neg \phi}{\phi \Rightarrow \psi} \text{ IR}
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Example 3

Given p and $\neg p$, prove q .

1. p Premise
2. $\neg p$ Premise

$$\begin{array}{c}
 \frac{\phi \Rightarrow \psi}{\frac{\phi}{\psi}} \text{ IE} \quad \frac{\psi}{\frac{\phi \Rightarrow \psi}{\phi \Rightarrow \psi}} \text{ IC} \quad \frac{(\phi \Rightarrow (\psi \Rightarrow \chi))}{(\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi)} \text{ ID} \quad \frac{\neg \psi \Rightarrow \neg \phi}{\phi \Rightarrow \psi} \text{ IR}
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Example 3

Given p and $\neg p$, prove q .

- | | |
|--------------------------------|---------|
| 1. p | Premise |
| 2. $\neg p$ | Premise |
| 3. $\neg q \Rightarrow \neg p$ | IC: 2 |

$$\begin{array}{c}
 \frac{\phi \Rightarrow \psi}{\frac{\phi}{\psi}} \text{ IE} \quad \frac{\psi}{\phi \Rightarrow \psi} \text{ IC} \quad \frac{(\phi \Rightarrow (\psi \Rightarrow \chi))}{(\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi)} \text{ ID} \quad \frac{\neg \psi \Rightarrow \neg \phi}{\phi \Rightarrow \psi} \text{ IR}
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|--------------------------------|---------|
| 1. p | Premise |
| 2. $\neg p$ | Premise |
| 3. $\neg q \Rightarrow \neg p$ | IC: 2 |
| 4. $p \Rightarrow q$ | IR: 3 |

$$\begin{array}{c}
 \frac{\phi \Rightarrow \psi}{\frac{\phi}{\psi}} \text{ IE} \quad \frac{\psi}{\phi \Rightarrow \psi} \text{ IC} \quad \frac{(\phi \Rightarrow (\psi \Rightarrow \chi))}{(\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi)} \text{ ID} \quad \frac{\neg \psi \Rightarrow \neg \phi}{\phi \Rightarrow \psi} \text{ IR}
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- | | |
|--------------------------------|----------|
| 1. p | Premise |
| 2. $\neg p$ | Premise |
| 3. $\neg q \Rightarrow \neg p$ | IC: 2 |
| 4. $p \Rightarrow q$ | IR: 3 |
| 5. q | IE: 4, 1 |

$$\begin{array}{c}
 \frac{\phi \Rightarrow \psi}{\frac{\phi}{\psi}} \text{ IE} \quad \frac{\psi}{\phi \Rightarrow \psi} \text{ IC} \quad \frac{(\phi \Rightarrow (\psi \Rightarrow \chi))}{(\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi)} \text{ ID} \quad \frac{\neg \psi \Rightarrow \neg \phi}{\phi \Rightarrow \psi} \text{ IR}
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Example 4

Given $\neg\neg p$, prove p .

1. $\neg\neg p$ Premise

$$\frac{\phi \Rightarrow \psi}{\frac{\phi}{\psi} \text{ IE}} \quad \frac{\psi}{\phi \Rightarrow \psi} \text{ IC} \quad \frac{(\phi \Rightarrow (\psi \Rightarrow \chi))}{(\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi)} \text{ ID} \quad \frac{\neg\psi \Rightarrow \neg\phi}{\phi \Rightarrow \psi} \text{ IR}$$

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Example 4

Given $\neg\neg p$, prove p .

1. $\neg\neg p$ Premise

2. $\neg\neg\neg\neg p \Rightarrow \neg\neg p$ IC: 1

$$\frac{\phi \Rightarrow \psi}{\frac{\phi}{\psi} \text{ IE}} \quad \frac{\psi}{\phi \Rightarrow \psi} \text{ IC} \quad \frac{(\phi \Rightarrow (\psi \Rightarrow \chi))}{(\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi)} \text{ ID} \quad \frac{\neg\psi \Rightarrow \neg\phi}{\phi \Rightarrow \psi} \text{ IR}$$

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Example 4

Given $\neg\neg p$, prove p .

1. $\neg\neg p$ Premise

2. $\neg\neg\neg\neg p \Rightarrow \neg\neg p$ IC: 1

3. $\neg p \Rightarrow \neg\neg\neg p$ IR: 2

4. $\neg\neg p \Rightarrow p$ IR: 3

$$\frac{\phi \Rightarrow \psi}{\frac{\phi}{\psi} \text{ IE}} \quad \frac{\psi}{\phi \Rightarrow \psi} \text{ IC} \quad \frac{(\phi \Rightarrow (\psi \Rightarrow \chi))}{(\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi)} \text{ ID} \quad \frac{\neg\psi \Rightarrow \neg\phi}{\phi \Rightarrow \psi} \text{ IR}$$

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Example 4

Given $\neg\neg p$, prove p .

- | | |
|--|----------|
| 1. $\neg\neg p$ | Premise |
| 2. $\neg\neg\neg p \Rightarrow \neg\neg p$ | IC: 1 |
| 3. $\neg p \Rightarrow \neg\neg\neg p$ | IR: 2 |
| 4. $\neg\neg p \Rightarrow p$ | IR: 3 |
| 5. p | IE: 4, 1 |

$$\begin{array}{c}
 \frac{\phi \Rightarrow \psi}{\psi} \text{ IE} \\
 \frac{\psi}{\phi \Rightarrow \psi} \text{ IC} \\
 \frac{(\phi \Rightarrow (\psi \Rightarrow \chi))}{(\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi)} \text{ ID} \\
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Axiom Schemas

Problem

Valid sentences are always true. So we should be able to prove them with no premises whatsoever.

Fact: $(p \Rightarrow p)$ is a valid sentence.

Problem: Prove $(p \Rightarrow p)$ without any premises.

The rules of inference we have seen so far all require some premises to start. Hmm.

Direct Proof

A *direct proof* of a conclusion from a set of premises is a sequence of sentences terminating in the conclusion in which each item is either

(1) a premise,

(2) an instance of a valid schema, or

(3) the result of applying a rule of inference to earlier items in sequence.

A schema is an expression satisfying the grammatical rules of our language except for the occurrence of metavariables (written here as Greek letters) in place of various subparts of the expression. For example, the following expression is a schema with metavariables ϕ and ψ .

Valid Schemata

A *valid schema* is a sentence pattern denoting an infinite set of sentences, *all of which* are valid.

Examples:

Reflexivity: $(\phi \Rightarrow \phi)$

Negation Elimination: $(\neg \neg \phi \Rightarrow \phi)$

Negation Introduction: $(\phi \Rightarrow \neg \neg \phi)$

Disjunctive Tautology: $(\phi \vee \neg \phi)$

Non-Examples:

$(\phi \wedge \neg \phi)$

$(\phi \Rightarrow \neg \phi)$

Rules and Axiom Schemata

Axiom Schemata as "0-ary" Rules of Inference

$\phi \Rightarrow \phi$

$\phi \Rightarrow \phi$

Rules of Inference as Axiom Schemata

$\frac{\neg \psi \Rightarrow \neg \phi}{\phi \Rightarrow \psi}$

$(\neg \psi \Rightarrow \neg \phi) \Rightarrow (\phi \Rightarrow \psi)$

We Need Both

NB: We need at least one axiom schema / 0-ary rule of inference to produce proofs of valid sentences.

NB: We need at least one non-0-ary rule of inference to derive conclusions from premises.

Hilbert System

Rule of Inference (IE):

$$\frac{\phi \Rightarrow \psi \quad \phi}{\psi}$$

Theoretically interesting.

Practically useless.

Pedagogical value.

Axiom Schemas (due to Lukasiewicz):

IC: $\phi \Rightarrow (\psi \Rightarrow \phi)$

ID: $(\phi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow ((\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi))$

IR: $(\neg\psi \Rightarrow \neg\phi) \Rightarrow (\phi \Rightarrow \psi)$

NB: Any PL sentence can be converted algorithmically into a logically equivalent sentence involving just $\neg$ and $\Rightarrow$.

Alternatively, we can add schemas for the other operators (Kleene system).

Invented by Jan Lukasiewicz (inventor of Polish notation - parenthesis free).

Proof Systems

A proof system is a finite set of (usually valid) axiom schemata and (usually sound) rules of inference.

Examples:

Hilbert / Lukasiewicz

Frege

Mendelson

Meredith

A proof system is a finite set of (usually valid) axiom schemata and (usually sound) rules of inference.

Example from Before

Given $(p \Rightarrow q)$ and $(q \Rightarrow r)$, prove $(p \Rightarrow r)$.

- | | |
|--|----------|
| 1. $p \Rightarrow q$ | Premise |
| 2. $q \Rightarrow r$ | Premise |
| 3. $p \Rightarrow (q \Rightarrow r)$ | IC: 2 |
| 4. $(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$ | ID: 3 |
| 5. $(p \Rightarrow r)$ | IE: 4, 1 |

$\phi \Rightarrow \psi$			
$\frac{\phi}{\psi}$	$\frac{\psi}{\phi \Rightarrow \psi}$	$\frac{(\phi \Rightarrow (\psi \Rightarrow \chi))}{(\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi)}$	$\frac{\neg \psi \Rightarrow \neg \phi}{\phi \Rightarrow \psi}$
IE	IC	ID	IR

Here is another example. Whenever p is true, q is true. Whenever q is true, r is true. With these as premises, we can prove that, whenever p is true, r is true. On line 3, we use Implication Creation to derive $(p \Rightarrow (q \Rightarrow r))$. On line 4, we use Implication Distribution to distribute the implication in line 3. Finally, on line 5, we use Implication Elimination to produce the desired result.

Example - Transitivity

Given $(p \Rightarrow q)$ and $(q \Rightarrow r)$, prove $(p \Rightarrow r)$.

- | | |
|--|----------|
| 1. $p \Rightarrow q$ | Premise |
| 2. $q \Rightarrow r$ | Premise |
| 3. $(q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))$ | IC |
| 4. $(p \Rightarrow (q \Rightarrow r))$ | IE: 2, 3 |
| 5. $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$ | ID |
| 6. $(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$ | IE: 4, 5 |
| 7. $p \Rightarrow r$ | IE: 1, 6 |

$\phi \Rightarrow \psi$	IC: $\psi \Rightarrow (\psi \Rightarrow \phi)$
$\frac{\phi}{\psi}$	ID: $(\phi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow ((\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi))$
$\frac{\psi}{\psi}$	IR: $(\neg \psi \Rightarrow \neg \phi) \Rightarrow (\phi \Rightarrow \psi)$
IE	

Here is an example. Whenever p is true, q is true. Whenever q is true, r is true. With these as premises, we can prove that, whenever p is true, r is true.

Example 2 - Reflexivity

Prove $(p \Rightarrow p)$.

- | | |
|--|----------|
| 1. $p \Rightarrow (p \Rightarrow p)$ | IC |
| 2. $p \Rightarrow ((p \Rightarrow p) \Rightarrow p)$ | IC |
| 3. $p \Rightarrow ((p \Rightarrow p) \Rightarrow p) \Rightarrow ((p \Rightarrow (p \Rightarrow p)) \Rightarrow (p \Rightarrow p))$ | ID |
| 4. $(p \Rightarrow (p \Rightarrow p)) \Rightarrow (p \Rightarrow p)$ | IE: 2, 3 |
| 5. $(p \Rightarrow p)$ | IE: 1, 4 |

$\phi \Rightarrow \psi$	IC: $\psi \Rightarrow (\psi \Rightarrow \phi)$
$\frac{\phi}{\psi}$	ID: $(\phi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow ((\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi))$
$\frac{\psi}{\psi}$	IR: $(\neg \psi \Rightarrow \neg \phi) \Rightarrow (\phi \Rightarrow \psi)$
IE	

Example 3 - Inconsistency

Given p and $\neg p$, prove q .

- | | |
|--|----------|
| 1. p | Premise |
| 2. $\neg p$ | Premise |
| 3. $\neg p \Rightarrow (\neg q \Rightarrow \neg p)$ | IC |
| 4. $\neg q \Rightarrow \neg p$ | IE: 3, 2 |
| 5. $(\neg q \Rightarrow \neg p) \Rightarrow (p \Rightarrow q)$ | IR |
| 6. $p \Rightarrow q$ | IE: 5, 4 |
| 7. q | IE: 6, 1 |

$\phi \Rightarrow \psi$	IC: $\phi \Rightarrow (\psi \Rightarrow \phi)$
$\frac{\phi}{\psi}$	ID: $(\phi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow ((\psi \Rightarrow \psi) \Rightarrow (\psi \Rightarrow \chi))$
IE	IR: $(\neg \psi \Rightarrow \neg \phi) \Rightarrow (\phi \Rightarrow \psi)$

Meredith System

Rule of Inference:

$$\frac{\phi \Rightarrow \psi}{\phi} \quad \psi$$

Axiom Schema:

$$(((\phi \Rightarrow \psi) \Rightarrow (\neg \chi \Rightarrow \neg \theta)) \Rightarrow \chi) \Rightarrow \tau) \Rightarrow ((\tau \Rightarrow \phi) \Rightarrow (\theta \Rightarrow \phi)))$$

Interesting. But what is even better is a set of axiom schemata that makes proofs easier to create.

Meredith System

Unfortunately:

Proofs hard to generate
Proofs hard to understand

Theoretically interesting.
Practically useless.
Minor pedagogical value.

Interesting. But what is even better is a set of axiom schemata that makes proofs easier to create.

Soundness and Completeness

Provability

If there exists a proof of a sentence ϕ from a set Δ of premises using the rules of inference in R , we say that ϕ is *provable* from Δ using R .

We usually write this as $\Delta \vdash_R \phi$, using the provability operator $\vdash$ (which is sometimes called *single turnstile*). (If the set of rules is clear from context, we usually drop the subscript, writing just $\Delta \vdash \phi$.)

$$\{p \Rightarrow q, q \Rightarrow r\} \vdash (p \Rightarrow r)$$

Let R be a set of rules of inference. If there exists a proof of a sentence ϕ from a set Δ of premises using the rules of inference in R , we say that ϕ is provable from Δ using R . We usually write this as $\Delta \vdash_R \phi$, using the provability operator $\vdash$ (which is sometimes called single turnstile). If the set of rules is clear from context, we usually drop the subscript, writing just $\Delta \vdash \phi$.

Note that the set of rules presented here is not powerful enough to prove everything that is entailed by a set of premises in Propositional Logic. There is no support for using or deducing negations or conjunctions or disjunctions or biconditionals. Even if we restrict ourselves to implications, we need more rules. While such rules of inference exist, they are a little complicated. For many people, it is easier to reason about implications using hypothetical reasoning.

Logical Entailment and Provability

A set of premises Δ *logically entails* a conclusion φ ($\Delta \models \varphi$) if and only if every interpretation that satisfies Δ also satisfies φ .

If there exists a proof of a sentence ϕ from a set Δ of premises using the rules of inference in R , we say that ϕ is *provable* from Δ using R (written $\Delta \vdash_R \phi$).

In talking about Logic, we now have two notions - logical entailment and provability. A set of premises logically entails a conclusion if and only if every truth assignment that satisfies the premises also satisfies the conclusion. A sentence is provable from a set of premises if and only if there is a finite proof of the conclusion from the premises. The concepts are quite different. One is based on truth assignments; the other is based on symbolic manipulation of expressions. Yet, for the proof systems we have been examining, they are closely related.

Soundness and Completeness

A proof system is *sound* if and only if every provable conclusion is logically entailed.

If $\Delta \vdash \phi$, then $\Delta \models \phi$.

A proof system is *complete* if and only if every logically entailed conclusion is provable.

If $\Delta \models \phi$, then $\Delta \vdash \phi$.

We say that a proof system is sound if and only if every provable conclusion is logically entailed. In other words, if $\Delta \vdash \phi$, then $\Delta \models \phi$. We say that a proof system is complete if and only if every logical conclusion is provable. In other words, if $\Delta \models \phi$, then $\Delta \vdash \phi$.

Hilbert System

Theorem: The Hilbert System is sound and complete for Propositional Logic.

$\Delta \vdash_{\text{Hilbert}} \phi$ if and only if $\Delta \models \phi$.

Translation: *The truth table method and the proof method succeed in exactly the same cases.*

NB: The Meredith System is also sound and complete.

$\Delta \vdash_{\text{Meredith}} \phi$ if and only if $\Delta \models \phi$.

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Significance

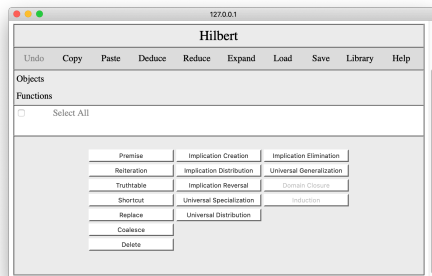
Proofs are often much smaller than the corresponding truth tables.

Finding a proof often takes fewer steps than the truth table method. (However, in the *worst* case, finding a proof may take just as many or even more steps.)

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Hilbert Online System

<http://logica.stanford.edu/documentation/hilbert.html>



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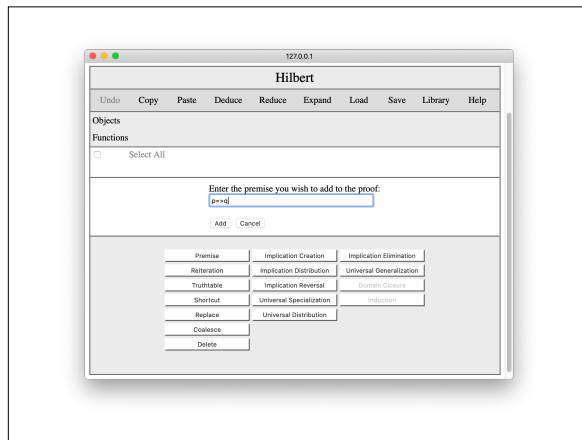
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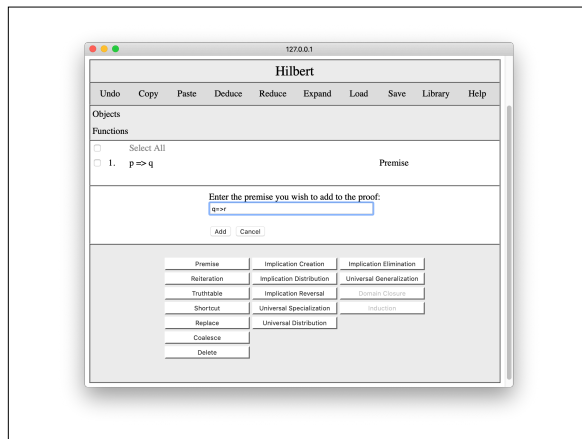


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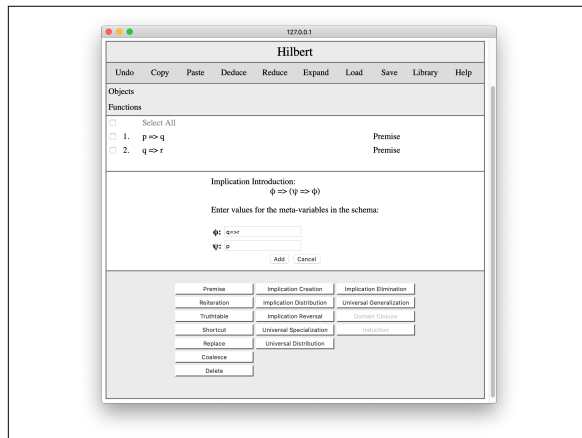
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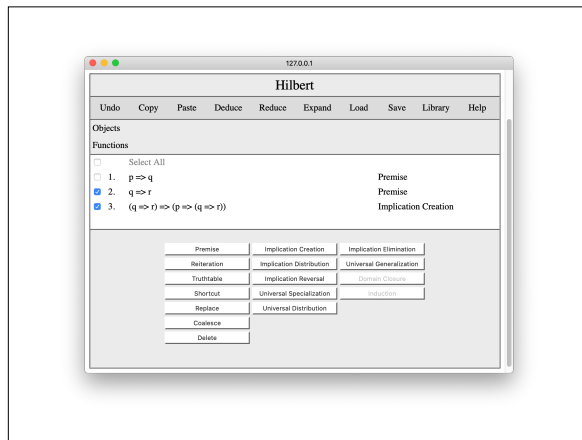
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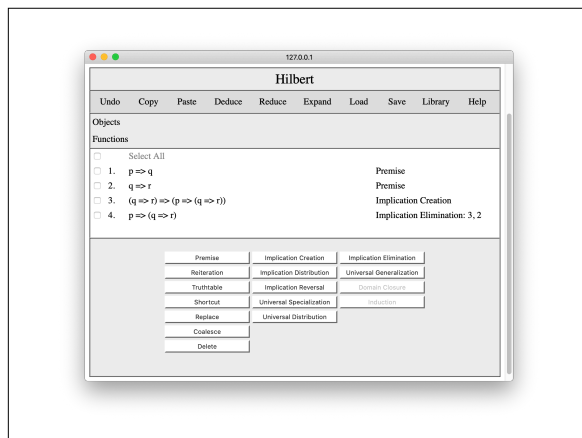
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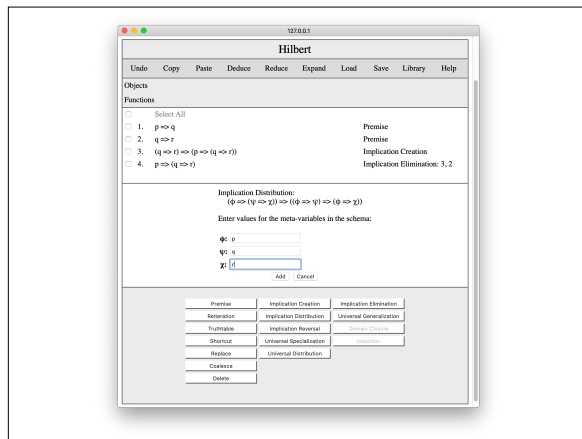
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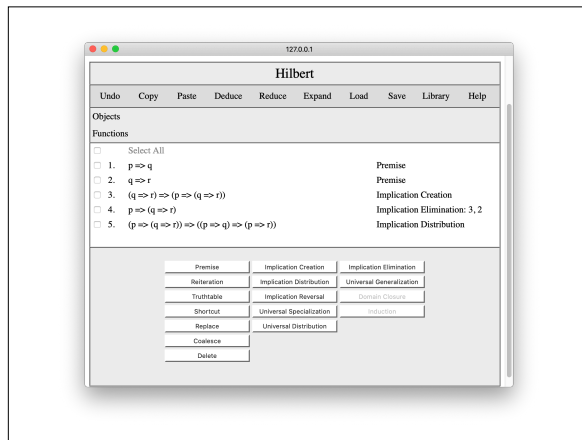
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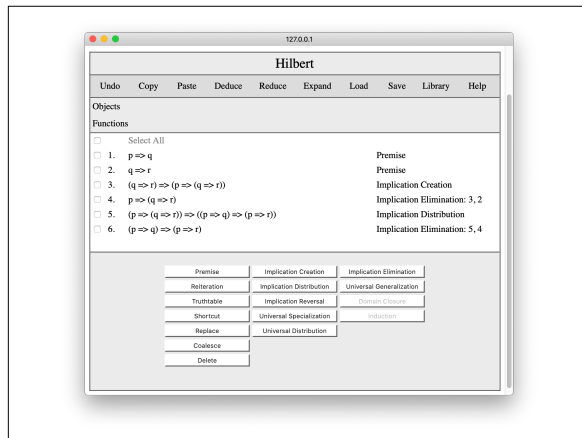
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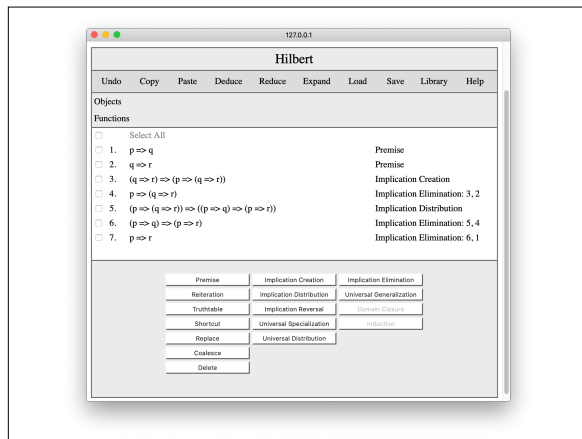
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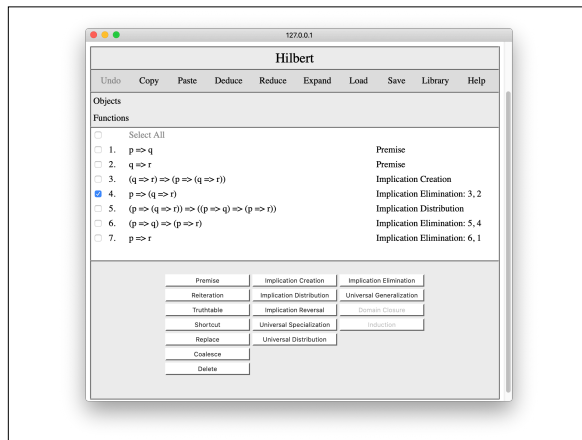
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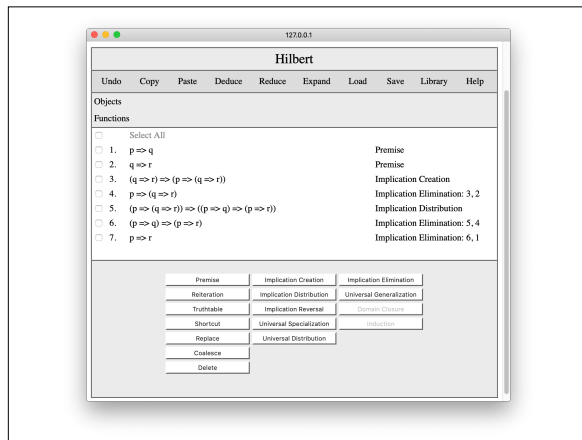
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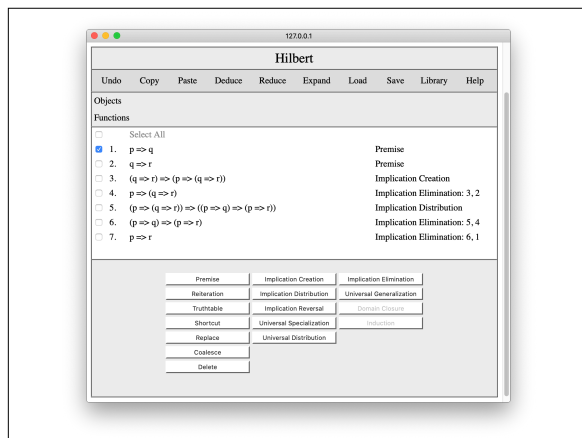
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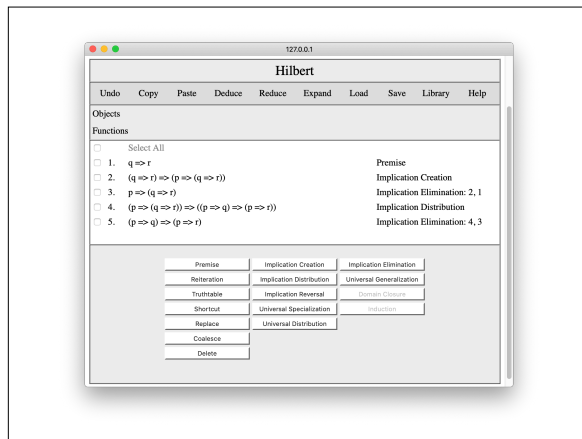
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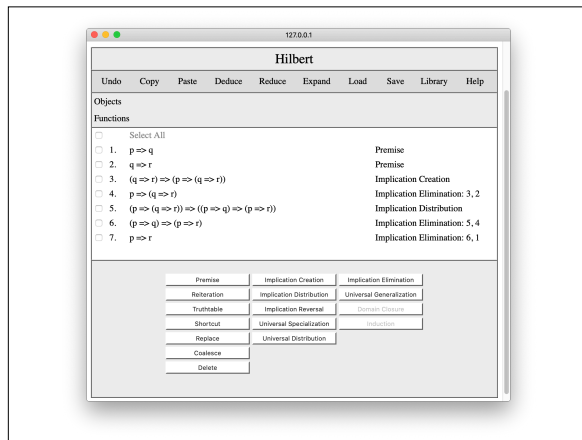
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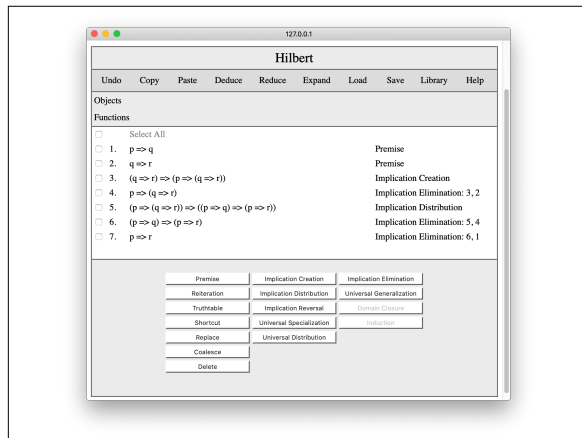
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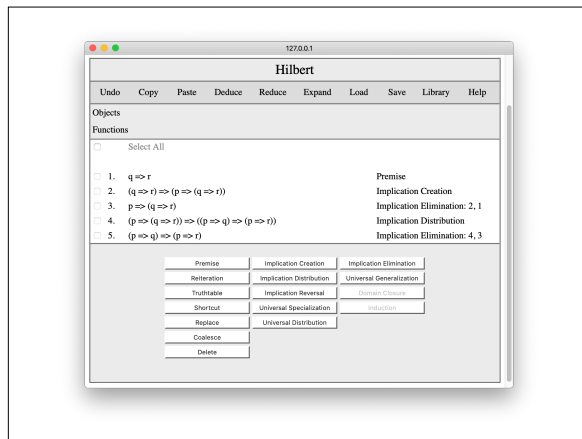
delete old version oops, undo

coalesce old version

copy and paste into different window

save and load
library
mention proof modification - substitute
mention proof merging - deduce, reduce, expand

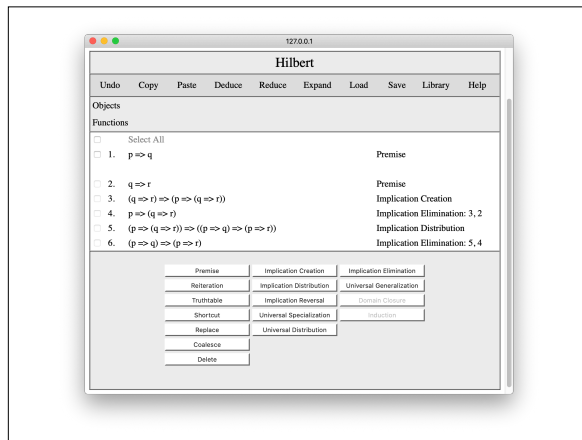
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undo check steps and deduce
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check step and use reduce
add $p \Rightarrow r$ via shortcut
check steps and click expand
substitution $s \Rightarrow t$



Point out documentation

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reiterate
undo
delete $p \Rightarrow q$ note removes dependents
change cursor and reinsert and reprove
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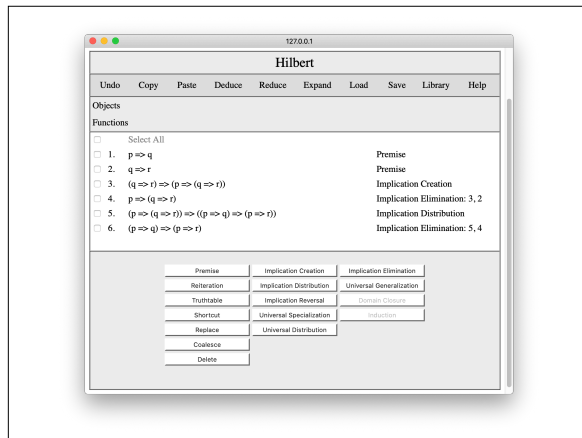
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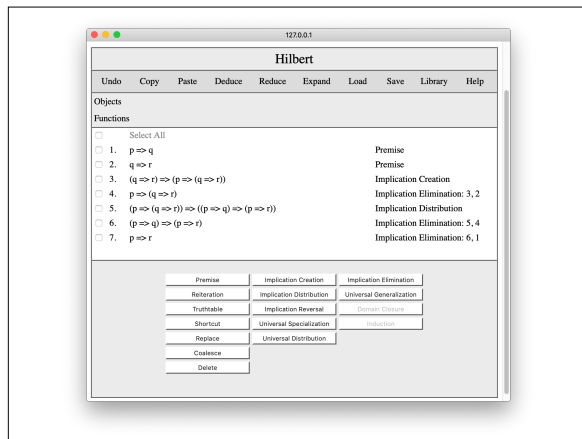
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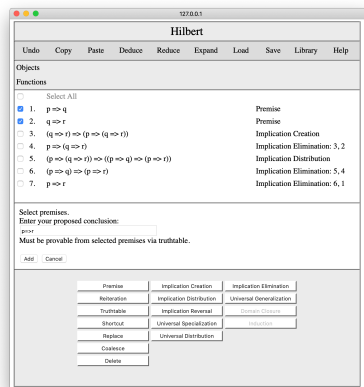
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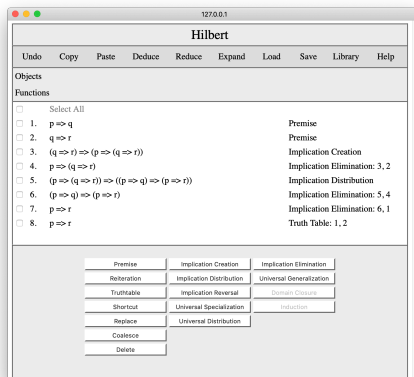
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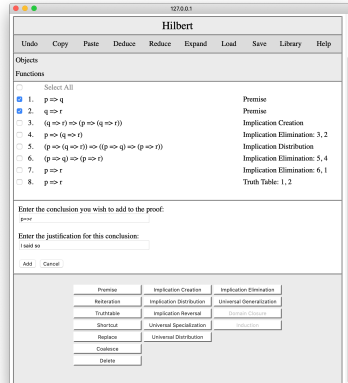
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From Hilbert to Fitch

Example

Example: Given $(p \Rightarrow q)$ and $(q \Rightarrow r)$, prove $(p \Rightarrow r)$.

1. $p \Rightarrow q$
2. $q \Rightarrow r$
3. $(q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))$
4. $(p \Rightarrow (q \Rightarrow r))$
5. $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$
6. $(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$
7. $p \Rightarrow r$

If there is a proof of ψ from Δ and ϕ , then there is a proof of $\phi \Rightarrow \psi$ from Δ .

Deduction Theorem

Deduction Theorem: $\Delta \models (\phi \Rightarrow \psi)$ if and only if $\Delta \cup \{\phi\} \models \psi$.

Corollary: $\Delta \vdash (\phi \Rightarrow \psi)$ if and only if $\Delta \cup \{\phi\} \vdash \psi$.

Problem: Given $(p \Rightarrow q)$ and $(q \Rightarrow r)$, prove $(p \Rightarrow r)$.

Equivalent: Given $(p \Rightarrow q)$ and $(q \Rightarrow r)$ and p , prove r .

If there is a proof of ψ from Δ and ϕ , then there is a proof of $\phi \Rightarrow \psi$ from Δ .

Deduction Theorem Example

Example: Given $(p \Rightarrow q)$ and $(q \Rightarrow r)$, prove $(p \Rightarrow r)$.

- | | |
|--|----------------------|
| 1. $p \Rightarrow q$ | 1. $p \Rightarrow q$ |
| 2. $q \Rightarrow r$ | 2. $q \Rightarrow r$ |
| 3. $(q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))$ | 3. p |
| 4. $(p \Rightarrow (q \Rightarrow r))$ | 4. q |
| 5. $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$ | 5. r |
| 6. $(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$ | |
| 7. $p \Rightarrow r$ | |

*Possible to derive long proof from short proof automatically.
Basis for the Fitch procedure.*

If there is a proof of ψ from Δ and ϕ , then there is a proof of $\phi \Rightarrow \psi$ from Δ .

