

Introduction to Logic

Natural Deduction

Michael Genesereth
Computer Science Department
Stanford University

Proof Systems

Popular Types of Proof Systems:

- Direct Proofs (Hilbert)
- Natural Deduction (Fitch)
- Refutation proofs (Resolution / Robinson)

Others:

Gentzen Systems
Sequent Calculi
and so forth

In our last lesson, we began our look at proof systems. In direct proof systems (which we saw last time), proofs take the form of sequences of sentences leading from premises to conclusions. In natural deduction systems (which we will see today), proofs have a tree-like structure - with subproofs nested within superproofs.

Example - Transitivity Proof

Given $(p \Rightarrow q)$ and $(q \Rightarrow r)$, prove $(p \Rightarrow r)$.

- | | |
|--|----------|
| 1. $p \Rightarrow q$ | Premise |
| 2. $q \Rightarrow r$ | Premise |
| 3. $(q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))$ | IC |
| 4. $(p \Rightarrow (q \Rightarrow r))$ | IE: 2, 3 |
| 5. $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$ | ID |
| 6. $(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$ | IE: 4, 5 |
| 7. $p \Rightarrow r$ | IE: 1, 6 |

How do we choose instances of IC and ID?

Here is an example. Whenever p is true, q is true. Whenever q is true, r is true. With these as premises, we can prove that, whenever p is true, r is true.

Transitivity - Related Proof

Given $(p \Rightarrow q)$ and $(q \Rightarrow r)$ and p , prove r .

1. $p \Rightarrow q$	Premise
2. $q \Rightarrow r$	Premise
3. p	Premise
4. q	IE: 1, 3
5. r	IE: 2, 4

Deduction Theorem

$\Delta \vdash (\varphi \Rightarrow \psi)$ if and only if $\Delta \cup \{\varphi\} \vdash \psi$

If there is a proof of ψ from Δ and φ , then there is a proof of $\varphi \Rightarrow \psi$ from Δ .

Deduction Theorem

Deduction Theorem: $\Delta \vdash (\varphi \Rightarrow \psi)$ if and only if $\Delta \cup \{\varphi\} \vdash \psi$.

$$\begin{array}{c} \{(p \Rightarrow q), (q \Rightarrow r)\} \vdash (p \Rightarrow r) \\ \text{if and only if} \\ \{(p \Rightarrow q), (q \Rightarrow r), p\} \vdash r \end{array}$$

There is a proof of $(p \Rightarrow r)$ from $\{(p \Rightarrow q), (q \Rightarrow r)\}$ if and only if there is a proof of r from $\{(p \Rightarrow q), (q \Rightarrow r), p\}$.

If there is a proof of ψ from Δ and φ , then there is a proof of $\varphi \Rightarrow \psi$ from Δ .

Transitivity - Hilbert Proof

Given $(p \Rightarrow q)$ and $(q \Rightarrow r)$, prove $(p \Rightarrow r)$.

- | | |
|--|----------|
| 1. $p \Rightarrow q$ | Premise |
| 2. $q \Rightarrow r$ | Premise |
| 3. $(q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))$ | IC |
| 4. $(p \Rightarrow (q \Rightarrow r))$ | IE: 2, 3 |
| 5. $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$ | ID |
| 6. $(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$ | IE: 4, 5 |
| 7. $p \Rightarrow r$ | IE: 1, 6 |

NB: This is a proof of $(p \Rightarrow r)$ from $(p \Rightarrow q)$ and $(q \Rightarrow r)$.

Recall our proof of transitivity of implication in the Hilbert system. Tricky because we needed to guess what we need in IC and ID in order to derive the desired answer.

Transitivity - Related Proof

Given $(p \Rightarrow q)$ and $(q \Rightarrow r)$ and p , prove r .

- | | |
|----------------------|----------|
| 1. $p \Rightarrow q$ | Premise |
| 2. $q \Rightarrow r$ | Premise |
| 3. p | Premise |
| 4. q | IE: 1, 3 |
| 5. r | IE: 2, 4 |

*NB: This is **not** itself a proof of $(p \Rightarrow r)$ from $(p \Rightarrow q)$ and $(q \Rightarrow r)$, but the deduction theorem tells us there **is** such a proof.*

Natural Deduction

Natural Deduction

Natural Deduction incorporates the deduction theorem in a new type of inference rule and an extended notion of proof.

More "natural" for most people.

No need for axiom schemata!!

Natural deduction simplifies matters by allowing one to make assumptions, prove some consequences, and thereby deduce implications involving the assumptions and the ultimate conclusions.

In natural deduction, there are three basic operations involved in creating useful subproofs - (1) making assumptions, (2) using ordinary rules of inference to derive conclusions, and (3) using structured rules of inference to derive conclusions outside of subproofs. Let's look at each of these operations in turn.

Essence of Natural Deduction

Making Assumptions

e.g. assume p

Using Assumptions

e.g. derive q

Discharging Assumptions leading to implications

e.g. conclude $p \Rightarrow q$

Natural deduction simplifies matters by allowing one to make assumptions, prove some consequences, and thereby deduce implications involving the assumptions and the ultimate conclusions.

In natural deduction, there are three basic operations involved in creating useful subproofs - (1) making assumptions, (2) using ordinary rules of inference to derive conclusions, and (3) using structured rules of inference to derive conclusions outside of subproofs. Let's look at each of these operations in turn.

Making Assumptions

In a natural deduction proof, it is permissible to make an arbitrary *assumption* in a nested proof.

NB: We can assume *anything we like*. We can assume *anything we like*. We can assume *anything we like*.

p
 $p \Rightarrow q$
 $p \vee \neg p$
 $p \wedge \neg p$

This is **okay** since (1) an assumption is used only within a subproof and (2) everything we prove in that subproof **depends** on that assumption (as we shall see in a moment).

Natural deduction simplifies matters by allowing one to make assumptions, prove some consequences, and thereby deduce implications involving the assumptions and the ultimate conclusions. Natural deduction proofs are usually shorter and simpler than direct proofs and so are favored by many for practical theorem proving.

As an example, consider the natural deduction proof of transitivity shown here. On line 3, we begin a subproof with the assumption that p is true. Note that p is not a premise in the overall problem. In a subproof, we can make whatever assumptions that we like. From p , we derive q using the premise on line 1; and, from that q , we prove r using the premise on line 2. That terminates the subproof. Finally, from this subproof, we derive $(p \Rightarrow r)$ in the outer proof. Given p , we can prove r ; and so we know $(p \Rightarrow r)$. The rule used in this case is called Implication Introduction, or **II** for short.

In a structured proof, it is permissible to make an arbitrary assumption in any subproof.

The assumptions need not be members of the initial premise set. Note that such assumptions cannot be used directly outside of the subproof, only as conditions in derived implications, so they do not contaminate the superproof or any unrelated subproofs. For example, in the proof we just saw, we used this assumption operation in the nested subproof even though p was not among the given premises.

Using Assumptions

Such assumptions can be used *within subproofs* to derive conclusions using ordinary rules of inference.

- | | | |
|----|-------------------|-------------------------------|
| 1. | $p \Rightarrow q$ | Premise |
| 2. | $q \Rightarrow r$ | Premise |
| 3. | p | Assumption |
| 4. | q | Implication Elimination: 1, 3 |
| 5. | r | Implication Elimination: 2, 4 |

NB: This is a proof of r *under the assumption* that p is true. It is *not* a proof of r from the premises alone.

In a structured proof, it is permissible to make an arbitrary assumption in any subproof. The assumptions need not be members of the initial premise set. Note that such assumptions cannot be used directly outside of the subproof, only as conditions in derived implications, so they do not contaminate the superproof or any unrelated subproofs.

For example, in the proof we just saw, we used this assumption operation in the nested subproof even though p was not among the given premises.

Discharging Assumptions

After exiting a subproof, we infer an **implication** with the assumption as antecedent and the conclusion as consequent.

- | | | |
|----|-------------------|--------------------------------|
| 1. | $p \Rightarrow q$ | Premise |
| 2. | $q \Rightarrow r$ | Premise |
| 3. | p | Assumption |
| 4. | q | Implication Elimination: 1, 3 |
| 5. | r | Implication Elimination: 2, 4 |
| 6. | $p \Rightarrow r$ | Implication Introduction: 3, 5 |

Everything we prove in a subproof depends on the assumption, and we capture that dependence in the derived implication.

For example, in the proof we just saw, we used this assumption operation in the nested subproof even though p was not among the given premises.

Terminology

A *structured rule of inference* is a pattern of reasoning consisting of one or more schemas, called *premises*, and one or more additional schemas, called *conclusions*, in which one of the premises is a condition of the form $\phi \vdash \psi$.

$$\frac{\phi \vdash \psi}{\phi \Rightarrow \psi}$$

Translation: If there is a subproof with assumption ϕ and conclusion ψ , then we conclude $\phi \Rightarrow \psi$ outside the subproof.

This new rule of inference is called *Implication Introduction*.

Correctly utilizing results derived in subproofs is the responsibility of a new type of rule of inference. Like an ordinary rule of inference, a structured rule of inference is a pattern of reasoning consisting of one or more premises and one or more conclusions. As before, the premises and conclusions can be schemas. However, the premises can also include conditions of the form $\phi \vdash \psi$. The rule in this case is called Implication Introduction, because it allows us to introduce new implications.

Terminology

A *structured proof* of a conclusion from a set of premises is a sequence of (possibly nested) sentences terminating in an occurrence of the conclusion at the top level of the proof. Each step in the proof must be either

- (1) a premise (at the top level),
- (2) an assumption, or
- (3) the result of applying an **ordinary rule of inference** or a **structured rule of inference** to earlier items in the sequence.

Finally, we define a structured proof of a conclusion from a set of premises to be a sequence of (possibly nested) sentences terminating in an occurrence of the conclusion at the top level of the proof. Each step in the proof must be either (1) a premise (at the top level) or an assumption (other than at the top level) or (2) the result of applying an ordinary or structured rule of inference to earlier items in the sequence (subject to the constraints given above).

Fitch

Fitch is a proof system that is particularly popular in the Logic community. It is as powerful as many other proof systems and is far simpler to use. Fitch achieves this simplicity through its support for structured proofs and its use of structured rules of inference in addition to ordinary rules of inference. Fitch has ten rules of inference in all. Nine of these are ordinary rules of inference. The other rule (Implication Introduction) is a structured rule of inference.

Implications

Implication Introduction (II):

$$\frac{\phi \vdash \psi}{\phi \Rightarrow \psi}$$

Implication Elimination (IE):

$$\frac{\phi \Rightarrow \psi \quad \phi}{\psi}$$

Implication Introduction is the structured rule we saw earlier. If, by assuming ϕ , we can derive ψ , then we can derive $(\phi \Rightarrow \psi)$. Implication Elimination is the first rule we saw at the start of this lecture.

Negations

Negation Introduction (NI):

$$\frac{\phi \Rightarrow \chi \quad \phi \Rightarrow \neg \chi}{\neg \phi}$$

Negation Elimination (NE):

$$\frac{\neg \neg \phi}{\phi}$$

Negation Introduction allows us to derive the negation of a sentence if it leads to a contradiction. If we believe $(\phi \Rightarrow \psi)$ and $(\phi \Rightarrow \neg \psi)$, then we can derive that ϕ is false. Negation Elimination allows us to delete double negatives.

Conjunctions

And Introduction (AI):

$$\frac{\phi \quad \psi}{\phi \wedge \psi}$$

And Elimination (AE):

$$\frac{\phi \wedge \psi}{\phi} \quad \frac{\phi \wedge \psi}{\psi}$$

And Introduction (shown below on the left) allows us to derive a conjunction from its conjuncts. If a proof contains sentences ϕ_1 through ϕ_n , then we can infer their conjunction. And Elimination (shown below on the right) allows us to derive conjuncts from a conjunction. If we have the conjunction of ϕ_1 through ϕ_n , then we can infer any of the conjuncts.

Disjunctions

Or Introduction (OI):

$$\frac{\phi}{\phi \vee \psi}$$

Or Elimination (OE):

$$\frac{\begin{array}{l} \phi \vee \psi \\ \phi \Rightarrow \chi \\ \psi \Rightarrow \chi \end{array}}{\chi}$$

Or Introduction allows us to infer an arbitrary disjunction so long as at least one of the disjuncts is already in the proof. Or Elimination is a little more complicated than And Elimination. Since we do not know which of the disjuncts is true, we cannot just drop the $\vee$. However, if we know that every disjunct entails some sentence, then we can infer that sentence even if we do not know which disjunct is true.

Equivalences / Biconditionals

Biconditional Introduction (BI):

$$\frac{\begin{array}{l} \phi \Rightarrow \psi \\ \psi \Rightarrow \phi \end{array}}{\phi \Leftrightarrow \psi}$$

Biconditional Elimination (BE):

$$\frac{\phi \Leftrightarrow \psi}{\begin{array}{l} \phi \Rightarrow \psi \\ \psi \Rightarrow \phi \end{array}}$$

Biconditional Introduction allows us to deduce a biconditional from an implication and its inverse. Biconditional Elimination goes the other way, allowing us to deduce two implications from a single biconditional.

Examples

Mary and Pat and Quincy

If Mary loves Pat, then Mary loves Quincy. If it is Monday, then Mary loves Pat or Quincy. If it is Monday, does Mary love Quincy?

Premises:

$$p \Rightarrow q$$
$$m \Rightarrow p \vee q$$

Goal:

$$m \Rightarrow q$$

As another example, recall the example of formal reasoning introduced in Chapter 1. We said that, whenever we have two rules in which the left hand side of one contains a proposition constant that occurs on the right hand side of the other, then we can cancel those constants and deduce a new rule by combining the remaining constants on the left hand sides of both rules and the remaining constants on the right hand sides of both rules. As it turns out, this is just Propositional Resolution.

Proof

Goal: $m \Rightarrow q$

1. $p \Rightarrow q$ Premise
2. $m \Rightarrow p \vee q$ Premise

As another example, recall the example of formal reasoning introduced in Chapter 1. We said that, whenever we have two rules in which the left hand side of one contains a proposition constant that occurs on the right hand side of the other, then we can cancel those constants and deduce a new rule by combining the remaining constants on the left hand sides of both rules and the remaining constants on the right hand sides of both rules. As it turns out, this is just Propositional Resolution.

Proof

Goal: $m \Rightarrow q$

1. $p \Rightarrow q$ Premise
2. $m \Rightarrow p \vee q$ Premise
3. $\downarrow m$ Assumption

As another example, recall the example of formal reasoning introduced in Chapter 1. We said that, whenever we have two rules in which the left hand side of one contains a proposition constant that occurs on the right hand side of the other, then we can cancel those constants and deduce a new rule by combining the remaining constants on the left hand sides of both rules and the remaining constants on the right hand sides of both rules. As it turns out, this is just Propositional Resolution.

Proof		
Goal: $m \Rightarrow q$		
1.	$p \Rightarrow q$	Premise
2.	$m \Rightarrow p \vee q$	Premise
3.	$\left m \right.$	Assumption
4.	$\left p \vee q \right.$	Implication Elimination: 2, 3

As another example, recall the example of formal reasoning introduced in Chapter 1. We said that, whenever we have two rules in which the left hand side of one contains a proposition constant that occurs on the right hand side of the other, then we can cancel those constants and deduce a new rule by combining the remaining constants on the left hand sides of both rules and the remaining constants on the right hand sides of both rules. As it turns out, this is just Propositional Resolution.

Proof		
Goal: $m \Rightarrow q$		
1.	$p \Rightarrow q$	Premise
2.	$m \Rightarrow p \vee q$	Premise
3.	$\left m \right.$	Assumption
4.	$\left p \vee q \right.$	Implication Elimination: 2, 3
5.	$\left q \right.$	Assumption
6.	$\left q \Rightarrow q \right.$	Implication Introduction: 5, 5

As another example, recall the example of formal reasoning introduced in Chapter 1. We said that, whenever we have two rules in which the left hand side of one contains a proposition constant that occurs on the right hand side of the other, then we can cancel those constants and deduce a new rule by combining the remaining constants on the left hand sides of both rules and the remaining constants on the right hand sides of both rules. As it turns out, this is just Propositional Resolution.

Proof		
Goal: $m \Rightarrow q$		
1.	$p \Rightarrow q$	Premise
2.	$m \Rightarrow p \vee q$	Premise
3.	$\left m \right.$	Assumption
4.	$\left p \vee q \right.$	Implication Elimination: 2, 3
5.	$\left q \right.$	Assumption
6.	$\left q \Rightarrow q \right.$	Implication Introduction: 5, 5
7.	$\left q \right.$	Or Elimination: 4, 1, 6

As another example, recall the example of formal reasoning introduced in Chapter 1. We said that, whenever we have two rules in which the left hand side of one contains a proposition constant that occurs on the right hand side of the other, then we can cancel those constants and deduce a new rule by combining the remaining constants on the left hand sides of both rules and the remaining constants on the right hand sides of both rules. As it turns out, this is just Propositional Resolution.

Proof

Goal: $m \Rightarrow q$

- | | | |
|----|--------------------------|--------------------------------|
| 1. | $p \Rightarrow q$ | Premise |
| 2. | $m \Rightarrow p \vee q$ | Premise |
| 3. | m | Assumption |
| 4. | $p \vee q$ | Implication Elimination: 2, 3 |
| 5. | $\perp q$ | Assumption |
| 6. | $q \Rightarrow q$ | Implication Introduction: 5, 5 |
| 7. | q | Or Elimination: 4, 1, 6 |
| 8. | $m \Rightarrow q$ | Implication Introduction: 3, 7 |

As another example, recall the example of formal reasoning introduced in Chapter 1. We said that, whenever we have two rules in which the left hand side of one contains a proposition constant that occurs on the right hand side of the other, then we can cancel those constants and deduce a new rule by combining the remaining constants on the left hand sides of both rules and the remaining constants on the right hand sides of both rules. As it turns out, this is just Propositional Resolution.

Reflexivity - Hilbert Proof

Prove $(p \Rightarrow p)$.

- | | | |
|----|---|----------|
| 1. | $p \Rightarrow (p \Rightarrow p)$ | IC |
| 2. | $p \Rightarrow ((p \Rightarrow p) \Rightarrow p)$ | IC |
| 3. | $p \Rightarrow ((p \Rightarrow p) \Rightarrow p) \Rightarrow ((p \Rightarrow (p \Rightarrow p)) \Rightarrow (p \Rightarrow p))$ | ID |
| 4. | $(p \Rightarrow (p \Rightarrow p)) \Rightarrow (p \Rightarrow p)$ | IE: 2, 3 |
| 5. | $(p \Rightarrow p)$ | IE: 1, 4 |

Reflexivity - Fitch Proof

Prove $(p \Rightarrow p)$.

- | | | |
|----|-------------------|--------------------------------|
| 1. | $\perp p$ | Assumption |
| 2. | $p \Rightarrow p$ | Implication Introduction: 1, 1 |

Negation Elimination - Hilbert Proof

Prove $(\neg\neg p \Rightarrow p)$.

1. $\neg\neg p \Rightarrow (\neg\neg\neg p \Rightarrow \neg\neg p)$ IC
2. $(\neg\neg\neg p \Rightarrow \neg\neg p) \Rightarrow (\neg p \Rightarrow \neg\neg p)$ IR
3. $\neg\neg p \Rightarrow (\neg p \Rightarrow \neg\neg p)$ Transitivity: 1, 2
4. $(\neg p \Rightarrow \neg\neg p) \Rightarrow (\neg\neg p \Rightarrow p)$ IR
5. $\neg\neg p \Rightarrow (\neg\neg p \Rightarrow p)$ Transitivity: 3, 4
6. $(\neg\neg p \Rightarrow (\neg\neg p \Rightarrow p)) \Rightarrow$
 $((\neg\neg p \Rightarrow \neg\neg p) \Rightarrow (\neg\neg p \Rightarrow p))$ ID
7. $(\neg\neg p \Rightarrow \neg\neg p) \Rightarrow (\neg\neg p \Rightarrow p)$ IE: 6, 5
8. $\neg\neg p \Rightarrow \neg\neg p$ Reflexivity
9. $\neg\neg p \Rightarrow p$ IE: 7, 8

Negation Elimination - Fitch Proof

Prove $(\neg\neg p \Rightarrow p)$.

1. $\neg\neg p$ Assumption
2. p Negation Elimination: 1
3. $\neg\neg p \Rightarrow p$ Implication Introduction: 1, 2

Our Transitivity Example

1. $p \Rightarrow q$ Premise
2. $q \Rightarrow r$ Premise
3. p Assumption
4. q Implication Elimination: 3, 1
5. r Implication Elimination: 4, 2
6. $p \Rightarrow r$ Implication Introduction: 3, 5

For example, in the structured proof we have been looking at, it is okay to apply Implication Elimination to 1 and 3. And it is okay to use Implication Elimination on lines 2 and 4.

Caveat

An ordinary rule of inference *applies* to a subproof at any level of nesting if and only if there is an instance of the rule in which all of the premises occur earlier in the **subproof** or in a **superproof of that subproof**.

Importantly, it is *not* permissible to apply an ordinary rule of inference to items that occur in other subproofs.

An ordinary rule of inference applies to a particular subproof of a structured proof if and only if there is an instance of the rule in which all of the premises occur earlier in the subproof or in some superproof of the subproof. Importantly, it is not permissible to use sentences in subproofs of that subproof or in other subproofs of its superproofs.

Bad Proof

- | | | | | |
|------|--|-------------------|--------------------------------|-------------------------------|
| 1. | | $p \Rightarrow q$ | Premise | |
| 2. | | $q \Rightarrow r$ | Premise | |
| 3. | | | p | Assumption |
| 4. | | | q | Implication Elimination: 1, 3 |
| 5. | | | r | Implication Elimination: 2, 4 |
| 6. | | $p \Rightarrow r$ | Implication Introduction: 3, 5 | |
| X 7. | | r | Implication Elimination: 2, 4 | X |

However, it is NOT acceptable to use a sentence from a subproof in applying an ordinary rule of inference in a superproof.

The last line of the malformed proof shown here gives an example of this. It is not permissible to use Implication Elimination as shown here because it uses a conclusion from a subproof as a premise in an application of an ordinary rule of inference in its superproof.

Bad Proof

- | | | | | |
|------|--|------------------------|--------------------------------|---------------------------------|
| 1. | | $p \Rightarrow q$ | Premise | |
| 2. | | $q \Rightarrow r$ | Premise | |
| 3. | | | p | Assumption |
| 4. | | | q | Implication Elimination: 1, 3 |
| 5. | | | r | Implication Elimination: 2, 4 |
| 6. | | $p \Rightarrow r$ | Implication Introduction: 3, 5 | |
| 7. | | | $\neg r$ | Assumption |
| X 8. | | | r | Implication Elimination: 2, 4 X |
| 9. | | $\neg r \Rightarrow r$ | Implication Introduction: 7, 8 | |

The malformed proof shown here is another example. Here, line 8 is illegal because line 4 is not in the current subproof or a superproof of this subproof.

Soundness and Completeness

Logical Entailment and Provability

A set of premises Δ *logically entails* a conclusion ϕ ($\Delta \models \phi$) if and only if every interpretation that satisfies Δ also satisfies ϕ .

If there exists a proof of a sentence ϕ from a set Δ of premises using the rules of inference in R , we say that ϕ is *provable* from Δ using R (written $\Delta \vdash_R \phi$).

In talking about Logic, we now have two notions - logical entailment and provability. A set of premises logically entails a conclusion if and only if every truth assignment that satisfies the premises also satisfies the conclusion. A sentence is provable from a set of premises if and only if there is a finite proof of the conclusion from the premises. The concepts are quite different. One is based on truth assignments; the other is based on symbolic manipulation of expressions. Yet, for the proof systems we have been examining, they are closely related.

Soundness and Completeness

A proof system is *sound* if and only if every provable conclusion is logically entailed.

If $\Delta \vdash \phi$, then $\Delta \models \phi$.

A proof system is *complete* if and only if every logically entailed conclusion is provable.

If $\Delta \models \phi$, then $\Delta \vdash \phi$.

We say that a proof system is sound if and only if every provable conclusion is logically entailed. In other words, if $\Delta \vdash \phi$, then $\Delta \models \phi$. We say that a proof system is complete if and only if every logical conclusion is provable. In other words, if $\Delta \models \phi$, then $\Delta \vdash \phi$.

Fitch

Theorem: Fitch is sound and complete for Propositional Logic.

$\Delta \models \phi$ if and only if $\Delta \vdash_{\text{Fitch}} \phi$.

Upshot: The truth table method and the Fitch method succeed in exactly the same cases!

The Fitch system is sound and complete. In other words, for this system, logical entailment and provability are identical. An arbitrary set of sentences Δ logically entails an arbitrary sentence ϕ if and only if ϕ is provable from Δ using Fitch.

The upshot of this result is significant. On large problems, the proof method often takes fewer steps than the truth table method. (Disclaimer: In the worst case, the proof method may take just as many or more steps to find an answer as the truth table method.) Moreover, proofs are usually much smaller than the corresponding truth tables. So writing an argument to convince others does not take as much space.

Comparative Power

*Fitch can do anything Hilbert can do!**⁺_#

** No axiom schemata necessary.
+ More intuitive than Hilbert.
Proofs are usually shorter.*

Implication Creation

Prove $(p \Rightarrow (q \Rightarrow p))$.

- | | | |
|----|-----------------------------------|-------------|
| 1. | p | Assumption |
| 2. | q | Assumption |
| 3. | p | Reiteration |
| 4. | $q \Rightarrow p$ | II: 2, 3 |
| 5. | $p \Rightarrow (q \Rightarrow p)$ | II: 1, 4 |

Implication Distribution

Prove $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$.

- | | | |
|----|---|------------|
| 1. | $p \Rightarrow (q \Rightarrow r)$ | Assumption |
| 2. | $p \Rightarrow q$ | Assumption |
| 3. | p | Assumption |
| 4. | q | IE: 2, 3 |
| 5. | $q \Rightarrow r$ | IE: 1, 3 |
| 6. | r | IE: 5, 4 |
| 7. | $p \Rightarrow r$ | II: 3, 6 |
| 8. | $(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$ | II: 2, 7 |
| 9. | $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$ | II: 1, 8 |

Implication Reversal

Prove $(\neg q \Rightarrow \neg p) \Rightarrow (p \Rightarrow q)$.

- | | | |
|----|---|-------------|
| 1. | $\neg q \Rightarrow \neg p$ | Assumption |
| 2. | p | Assumption |
| 3. | $\neg q$ | Assumption |
| 4. | p | Reiteration |
| 5. | $\neg q \Rightarrow p$ | II: 3, 4 |
| 6. | $\neg \neg q$ | NI: 5, 1 |
| 7. | q | NE: 6 |
| 8. | $p \Rightarrow q$ | II: 2, 7 |
| 9. | $(\neg q \Rightarrow \neg p) \Rightarrow (p \Rightarrow q)$ | II: 1, 8 |

Practical Matters

Theorem Proving Requires Search

Good News: There *is* an algorithm for determining logical entailment in Propositional Logic (using Truth tables).

Bad News: Truth tables can be *very* large.

Good News: There *are* algorithms for finding Propositional Logic proofs using Hilbert and Fitch.

Bad News: These algorithms can be expensive. *Theorem proving requires search.*

Good News: Proofs can *often* be found quickly.

Good News: Proofs are *usually* smaller than truth tables.

Reasoning Tips

Tip 1: Given a *goal* $(\phi \wedge \psi)$, prove ϕ , prove ψ , and then use And Introduction to derive $(\phi \wedge \psi)$.

1. $p \Rightarrow q$	Premise
2. $p \Rightarrow r$	Premise
3. p	Premise
4. q	IE: 1, 3
5. r	IE: 2, 3
6. $q \wedge r$	AI: 4, 5

Reasoning Tips

Tip 2: Given a *goal* $(\phi \vee \psi)$, try to prove ϕ *or* prove ψ (only one is needed), then use Or Introduction to disjoin with *anything else*.

1. $p \wedge q$	Premise
2. p	AE: 1
3. q	AE: 1
4. $q \vee r$	OI: 3

Reasoning Tips

Tip 3: Given a *goal* $(\phi \Rightarrow \psi)$, it is often good to assume ϕ , prove ψ , and then use Implication Introduction.

- | | | |
|----|-------------------|----------------|
| 1. | q | Premise |
| 2. | p | Assumption |
| 3. | q | Reiteration: 1 |
| 4. | $p \Rightarrow q$ | II: 2, 3 |

Reasoning Tips

Tip 4: Given a *goal* of the form $(\phi \Leftrightarrow \psi)$, prove $(\phi \Rightarrow \psi)$, prove $(\psi \Rightarrow \phi)$, and then use Biconditional Introduction.

- | | |
|------|-----------------------|
| ... | ... |
| $m.$ | $p \Rightarrow q$ |
| ... | ... |
| $n.$ | $q \Rightarrow p$ |
| ... | ... |
| | $p \Leftrightarrow q$ |
- Biconditional Introduction: m, n

Reasoning Tips

Tip 5: Given a *goal* of the form $\neg\phi$, assume ϕ and derive the sentence $(\phi \Rightarrow \psi)$, assume ϕ again and derive the sentence $\neg\psi$ leading to $(\phi \Rightarrow \neg\psi)$, and use Negation Introduction to derive $\neg\phi$.

- | | | |
|----|------------------------|----------------|
| 1. | $p \Rightarrow q$ | Premise |
| 2. | $\neg q$ | Premise |
| 3. | p | Assumption |
| 4. | $\neg q$ | Reiteration: 2 |
| 5. | $p \Rightarrow \neg q$ | II: 3, 4 |
| 6. | $\neg p$ | NI: 1, 5 |

Reasoning Tips

Tip 6: Given a *goal* ϕ , assume $\neg\phi$, prove a contradiction, thereby deriving $\neg\neg\phi$, and then apply Negation Elimination to get ϕ .

- | | | |
|----|-----------------------------|----------------|
| 1. | $\neg p \Rightarrow q$ | Premise |
| 2. | $\neg q$ | Premise |
| 3. | $\neg p$ | Assumption |
| 4. | $\neg q$ | Reiteration: 2 |
| 5. | $\neg p \Rightarrow \neg q$ | II: 3, 4 |
| 6. | $\neg\neg p$ | NI: 1, 5 |
| 7. | p | NE: 6 |

Reasoning Tips

Tip 7: Given a *premise* $(\phi \wedge \psi)$, consider splitting it into its constituent conjuncts.

- | | | |
|----|--------------|---------|
| 1. | $p \wedge q$ | Premise |
| 2. | p | AE: 1 |
| 3. | q | AE: 1 |
| 4. | $q \vee r$ | OI: 3 |

Reasoning Tips

Tip 8: Given a *premise* $(\phi \Leftrightarrow \psi)$, consider splitting it into its constituent implications.

- | | | |
|----|-----------------------|---------|
| 1. | $p \Leftrightarrow q$ | Premise |
| 2. | $p \Rightarrow q$ | BE: 1 |
| 3. | $q \Rightarrow p$ | BE: 1 |

Reasoning Tips

Tip 9: Given a *premise* of the form $(\phi \Rightarrow \psi)$ and a goal ψ , try proving ϕ and then use Implication Elimination to derive ψ .

- | | |
|-------------------------------|----------|
| 1. $p \wedge q \Rightarrow r$ | Premise |
| 2. p | Premise |
| 3. q | Premise |
| 4. $p \wedge q$ | AI: 2 3 |
| 5. r | IE: 1, 4 |

Reasoning Tips

Tip 10: Given a *premise* $(\phi \vee \psi)$ and a goal χ , try proving $(\phi \Rightarrow \chi)$ and $(\psi \Rightarrow \chi)$ and use Or Elimination to derive χ .

- | | |
|----------------------|-------------|
| 1. $p \vee q$ | Premise |
| 2. $p \Rightarrow q$ | Premise |
| 3. $ q$ | Assumption |
| 4. $q \Rightarrow q$ | II: 3, 3 |
| 5. q | OE: 1, 2, 4 |

Fitch Online System

Course Website

<http://logica.stanford.edu>

Example

- | | | |
|----|---|--------------------------------|
| 1. | $p \Rightarrow q$ | Premise |
| 2. | $m \Rightarrow p \vee q$ | Premise |
| 3. | $\left \begin{array}{l} m \end{array} \right.$ | Assumption |
| 4. | $\left \begin{array}{l} p \vee q \end{array} \right.$ | Implication Elimination: 2, 3 |
| 5. | $\left \begin{array}{l} \bot q \end{array} \right.$ | Assumption |
| 6. | $\left \begin{array}{l} q \Rightarrow q \end{array} \right.$ | Implication Introduction: 5, 5 |
| 7. | $\left \begin{array}{l} q \end{array} \right.$ | Or Elimination: 4, 1, 6 |
| 8. | $m \Rightarrow q$ | Implication Introduction: 3, 7 |

Given: $p \Rightarrow q$ and $m \Rightarrow p \vee q$
Prove: $m \Rightarrow q$

Example

- | | | |
|-----|--|---------------------------------|
| 1. | $p \Rightarrow r$ | Premise |
| 2. | $q \Rightarrow s$ | Premise |
| 3. | $p \vee q$ | Premise |
| 4. | $\left \begin{array}{l} p \end{array} \right.$ | Assumption |
| 5. | $\left \begin{array}{l} r \end{array} \right.$ | Implication Elimination: 1, 4 |
| 6. | $\left \begin{array}{l} r \vee s \end{array} \right.$ | Or Introduction: 5 |
| 7. | $p \Rightarrow r \vee s$ | Implication Introduction: 4, 6 |
| 8. | $\left \begin{array}{l} q \end{array} \right.$ | Assumption |
| 9. | $\left \begin{array}{l} s \end{array} \right.$ | Implication Elimination: 2, 9 |
| 10. | $\left \begin{array}{l} r \vee s \end{array} \right.$ | Or Introduction: 9 |
| 11. | $q \Rightarrow r \vee s$ | Implication Introduction: 8, 10 |
| 12. | $r \vee s$ | Or Elimination: 3, 7, 11 |

Given: $p \Rightarrow r$, $q \Rightarrow s$, and $p \vee q$
Prove: $r \vee s$



Example

Coin Game

Heads, you win. Tails, I lose. Prove that you win.

- | | | |
|----|----------------------------|---------|
| 1 | $h \Rightarrow y$ | Premise |
| 2. | $t \Rightarrow \neg m$ | Premise |
| 3. | $h \vee t$ | Premise |
| 4 | $y \Leftrightarrow \neg m$ | Premise |

Coin Game

Heads, you win. Tails, I lose. Prove that you win.

- | | | |
|----|----------------------------|---------|
| 1 | $h \Rightarrow y$ | Premise |
| 2. | $t \Rightarrow \neg m$ | Premise |
| 3. | $h \vee t$ | Premise |
| 4 | $y \Leftrightarrow \neg m$ | Premise |
| 5. | $y \Rightarrow \neg m$ | BE: 4 |
| 6. | $\neg m \Rightarrow y$ | BE: 4 |

Coin Game

Heads, you win. Tails, I lose. Prove that you win.

1.	$h \Rightarrow y$	Premise
2.	$t \Rightarrow \neg m$	Premise
3.	$h \vee t$	Premise
4.	$y \Leftrightarrow \neg m$	Premise
5.	$y \Rightarrow \neg m$	BE: 4
6.	$\neg m \Rightarrow y$	BE: 4
7.	$\mid t$	Assumption

Coin Game

Heads, you win. Tails, I lose. Prove that you win.

1.	$h \Rightarrow y$	Premise
2.	$t \Rightarrow \neg m$	Premise
3.	$h \vee t$	Premise
4.	$y \Leftrightarrow \neg m$	Premise
5.	$y \Rightarrow \neg m$	BE: 4
6.	$\neg m \Rightarrow y$	BE: 4
7.	$\mid t$	Assumption
8.	$\mid \neg m$	IE: 2, 7

Coin Game

Heads, you win. Tails, I lose. Prove that you win.

1.	$h \Rightarrow y$	Premise
2.	$t \Rightarrow \neg m$	Premise
3.	$h \vee t$	Premise
4.	$y \Leftrightarrow \neg m$	Premise
5.	$y \Rightarrow \neg m$	BE: 4
6.	$\neg m \Rightarrow y$	BE: 4
7.	$\mid t$	Assumption
8.	$\mid \neg m$	IE: 2, 7
9.	$\mid y$	IE: 6, 8

Coin Game

Heads, you win. Tails, I lose. Prove that you win.

1.	$h \Rightarrow y$	Premise
2.	$t \Rightarrow \neg m$	Premise
3.	$h \vee t$	Premise
4.	$y \Leftrightarrow \neg m$	Premise
5.	$y \Rightarrow \neg m$	BE: 4
6.	$\neg m \Rightarrow y$	BE: 4
7.	t	Assumption
8.	$\neg m$	IE: 2, 7
9.	y	IE: 6, 8
10.	$t \Rightarrow y$	II: 7, 9

Coin Game

Heads, you win. Tails, I lose. Prove that you win.

1.	$h \Rightarrow y$	Premise
2.	$t \Rightarrow \neg m$	Premise
3.	$h \vee t$	Premise
4.	$y \Leftrightarrow \neg m$	Premise
5.	$y \Rightarrow \neg m$	BE: 4
6.	$\neg m \Rightarrow y$	BE: 4
7.	t	Assumption
8.	$\neg m$	IE: 2, 7
9.	y	IE: 6, 8
10.	$t \Rightarrow y$	II: 7, 9
11.	y	OE: 3, 1, 10



Given:

$p \Rightarrow r$

$q \Rightarrow s$

plq

Prove: rls